§5.1 Local Maxima and Minima

A function $y = f(x)$ has a **local maximum** at a point when the $y$-value at that point is greater than at any other point in the immediate neighbourhood. There may be larger values somewhere else but standing at the local maximum the ground falls away in both directions.

A **local minimum** is a point where the $y$-value is less than at any other point in the immediate neighbourhood. The plural of “maximum” and “minimum” are “maxima” and “minima”.

If there’s a derivative at a local maximum or minimum it clearly must be zero because a positive or negative slope would mean that the curve is higher on one side or the other. But there’s also the possibility that a function has no derivative at the local maximum or minimum.

The following graph clearly has a local minimum at the vertex but at that point there’s no uniquely defined slope.

There’s no tangent at this minimum point and so it doesn’t make sense to talk of the slope there. But we won’t encounter this sort of situation very often so to find the local maxima and minima we simply put $\frac{dy}{dx} = 0$ and solve for $x$.

**Stationary points** are where the slope is zero or, in other words, $\frac{dy}{dx} = 0$. They include local maxima and minima, but here’s another possibility.

Here you see a point where the derivative is zero, but it’s neither a local maximum nor a local minimum. We call it a **stationary point of inflection**. So there are three types of stationary point: local maxima, local minima and stationary points of inflection. Usually when asked to find the stationary points you’ll be asked to classify them. This means to determine what type of stationary point they are.
Example 1: Find the stationary points of the function \( f(x) = x^3 - 3x + 2 \).

Solution: \( f'(x) = 3x^2 - 3 \). The stationary points occur when \( f'(x) = 0 \). We solve \( 3x^2 - 3 = 0 \) and get \( x = \pm 1 \). Since \( f(1) = 0 \) and \( f(-1) = 4 \), there are stationary points at \((-1, 4)\) and \((1, 0)\). We might guess that \((-1, 4)\) is a local maximum and \((1, 0)\) is a local minimum and we’d be right. But it’s very easy to guess wrongly with more complicated examples. We need a test.

§5.2 The Second Derivative Test

The easiest test to understand, and often the easiest to use, is the so-called “Second Derivative Test”. There are three possibilities for the value of \( \frac{d^2y}{dx^2} \) at a stationary point. It can be positive, negative or zero. And there are three types of stationary point: maximum, minimum and stationary point of inflection. It would be tempting to suppose that the three possibilities for the value of \( \frac{d^2y}{dx^2} \) correspond to three types of stationary point, but unfortunately it’s not quite that simple.

If \( \frac{d^2y}{dx^2} < 0 \) this means that the derivative of the derivative is negative, or in other words, the derivative is decreasing. Since it’s zero at the stationary point this means that the slope must be positive to the left of the point and negative to the right. Clearly the stationary point must be a local maximum.

If \( \frac{d^2y}{dx^2} > 0 \) this means that the derivative of the derivative is positive, or in other words, the derivative is increasing. Since it’s zero at the stationary point this means that the slope must be negative to the left of the point and positive to the right. Clearly the stationary point must be a local minimum.

So far, so good. Does this mean that if \( \frac{d^2y}{dx^2} = 0 \) we must have a stationary point of inflection? NO! We might have one. But equally we might have a local maximum or a local minimum. Consider these three examples

Example 2: \( y = x^3 \); \( \frac{dy}{dx} = 3x^2 \); \( \frac{d^2y}{dx^2} = 6x \). There’s a stationary point at \( x = 0 \) and at this point and \( \frac{d^2y}{dx^2} = 0 \). What does the graph of \( y = x^3 \) look like?

This is a clear case of a stationary point of inflection.

Example 3: \( y = x^4 \); \( \frac{dy}{dx} = 4x^3 \); \( \frac{d^2y}{dx^2} = 12x^2 \). There’s a stationary point at \( x = 0 \) and at this point and \( \frac{d^2y}{dx^2} = 0 \). What does the graph of \( y = x^4 \) look like?
There’s a zero second derivative again, but this time the stationary point is a local minimum.

**Example 4:** $y = -x^4; \quad \frac{dy}{dx} = -4x^3; \quad \frac{d^2y}{dx^2} = -12x^2$. There’s a stationary point at $x = 0$ and at this point $\frac{d^2y}{dx^2} = 0$. What does the graph of $y = -x^4$ look like?

![Graph of $y = -x^4$]

Again a zero second derivative again, but this time the stationary point is a local maximum. The moral of the story is that:

**If the second derivative is zero this tells you ABSOLUTELY NOTHING.**

| SECOND DERIVATIVE TEST | For a Stationary Point (where $\frac{dy}{dx} = 0$):
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Sign of $\frac{d^2y}{dx^2}$</strong></td>
<td><strong>Nature of Stationary Point</strong></td>
</tr>
<tr>
<td>+</td>
<td>Local MINIMUM</td>
</tr>
<tr>
<td>−</td>
<td>Local MAXIMUM</td>
</tr>
<tr>
<td>0</td>
<td>TEST FAILS !!</td>
</tr>
</tbody>
</table>

**Example 5:** Find the stationary points of $y = x^3 - 3x + 2$ and determine their nature.

**Solution:** $\frac{dy}{dx} = 3x^2 - 3$. As in example 1 we equate this to zero and solve. We then find the corresponding $y$ values and so find that the stationary points are $(-1, 4)$ and $(1, 0)$.

Differentiating again we get $\frac{d^2y}{dx^2} = 6x - 3$.

When $x = -1$, $\frac{d^2y}{dx^2} = -9 < 0$ so $(-1, 4)$ is a local maximum.

When $x = 1$, $\frac{d^2y}{dx^2} = 3 > 0$ so $(1, 0)$ is a local minimum.

**§5.3 The First Derivative Test**

Suppose we have a stationary point to $y = f(x)$ at $x = a$. We can determine the nature of this stationary point by examining the sign of $\frac{dy}{dx}$ immediately to the left of $a$ and immediately to the right.

If $\frac{dy}{dx}$ is negative for $x < a$ and $\frac{dy}{dx}$ is positive for $x > a$ then the graph comes down, levels out momentarily, and then climbs again. This is a clear case of a local minimum. But before going on to the other cases we’d better clarify what we mean. We’re referring only to points *immediately*
to the left of $x = a$ and immediately to the right of $x = a$. It doesn’t matter what the value of $\frac{dy}{dx}$ is further away. So long as, for some $b < a$ we have $\frac{dy}{dx} < 0$ for $b < x < a$ (that is, for $x$ between $b$ and $a$) and, for some $c > a$ we have $\frac{dy}{dx} > 0$ for $a < x < c$, then there’s a local minimum at $x = a$.

Now that’s all a bit of a mouthful. So instead we write:

If $\frac{dy}{dx} < 0$ for $x = a^-$ and $\frac{dy}{dx} > 0$ for $x = a^+$ then there’s a local minimum at $x = a$.

The phrase “$x = a^-$” is shorthand for “$x = a$ minus-a-little-bit” or, more precisely, what we said above: “for some $b < a$ we have $\frac{dy}{dx} < 0$ for $b < x < a$”.

Using this useful piece of shorthand we can identify four possibilities. By working out which one applies to a given example we can determine the nature of the stationary point.

<table>
<thead>
<tr>
<th>FIRST DERIVATIVE TEST</th>
</tr>
</thead>
<tbody>
<tr>
<td>For a Stationary Point (where $\frac{dy}{dx} = 0$):</td>
</tr>
<tr>
<td>Sign of $\frac{dy}{dx}$ for $x = a^-$</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>$-$</td>
</tr>
<tr>
<td>$+$</td>
</tr>
<tr>
<td>$-$</td>
</tr>
<tr>
<td>$+$</td>
</tr>
</tbody>
</table>

Now, rather than having to remember the above table (it’s so easy to get the cases mixed up), you should simply draw the appropriate sketch, as indicated. Of course the actual curve won’t be a series of straight lines as shown here. These are simply crude approximations to the curve to help you decide on the nature of the stationary point.

Draw a short horizontal straight line to denote a stationary point. Then investigate the sign of $\frac{dy}{dx}$ immediately to the left of the point and immediately to the right. Where $\frac{dy}{dx}$ is negative, draw a short downhill slope. Where $\frac{dy}{dx}$ is positive draw a short uphill slope. The resulting picture will then indicate the nature of the stationary point.

You may have wondered what happens if $\frac{dy}{dx}$ is zero for all points immediately to the left, or immediately to the right of a stationary point. In such cases we have a whole region of stationary points. For example the horizontal straight line $x = 2$ consists of nothing else but stationary points. And these are neither local maxima nor minima. But nor are they stationary points of inflection. These types of stationary points are less common and not very useful, so we don’t give them special names.

**Example 6:** Find nature of the stationary points of $y = x^3 - 3x + 2$ using the First Derivative Test.

**Solution:** Since $\frac{dy}{dx} = 3x^2 - 3 = 3(x - 1)(x + 1)$, the stationary points occur when $x = \pm 1$. 

68
We have three factors here for \( \frac{dy}{dx} \). The first, 3, is always positive.

Now when \( x = 1^- \) (a little bit less than 1), \( x - 1 \) is negative and \( x + 1 \) is positive. So \( \frac{dy}{dx} = + + = - \) (shorthand for “the derivative is the product of a positive number, a negative number and a positive number and so is a negative number”).

You need to remember that a positive number times a negative number is negative, while the product of two negative numbers is positive.

The critical factor for \( x \) near 1 is \( x - 1 \). This is the one that changes sign as we pass from one side of \( x = 1 \) to the other. The third factor is going to be round about 2 and so will be positive both for \( x = 1^- \) and \( x = 1^+ \). If \( x = 1^- \) it’s a bit less than 2 and if \( x = 1^+ \) it’s a bit bigger than 2. It doesn’t make much difference. A little bit less than 2 or a little bit bigger than 2 is still going to be positive.

Of course if \( x \) is a lot less than 1, say \( x = -2 \), then \( x + 1 \) is no longer positive. But remember that \( x = 1^- \) refers to points immediately to the left, not ones way out past other stationary points. So there’s no problem.

We write:

\[
\frac{dy}{dx} = 3x^2 - 3 = 3(x - 1)(x + 1)
\]

When \( x = 1^- \) then \( \frac{dy}{dx} = + + = - \)

When \( x = 1^+ \) then \( \frac{dy}{dx} = + + = + \)

We then transfer this information to the sketch:
and conclude that there’s a local minimum at \( x = 1 \).

Repeating this for \( x = -1 \) we get:

When \( x = -1^- \) then \( \frac{dy}{dx} = + - = + \)

When \( x = -1^+ \) then \( \frac{dy}{dx} = + + = - \)

We then transfer this information to the sketch:
and conclude that there is a local maximum at \( x = 1 \).

§5.4 Which Should You Use: The First or Second Derivative Test?

The Second Derivative Test is more straightforward than the First and should always be the one you think of first. However it suffers from a few serious disadvantages.

For a start, the Second Derivative Test requires you to differentiate twice. This isn’t much of a problem with the simple functions we’ve used so far but there are more complicated functions where it’s difficult enough to find the first derivative, and a nightmare to differentiate a second time. If you find yourself having to work hard to differentiate the derivative again you should consider using the First Derivative Test instead.

The other serious deficiency of the Second Derivative Test is that it sometimes shrugs its shoulders and says, “I don’t know!” This is when the second derivative is zero at the stationary point. Perhaps it’s a stationary point of inflection. On the other hand it might be a local maximum or minimum. Stumped! In such cases, whether you like it or not, you have to fall back on the First Derivative Test.

Example 7: Find the stationary points and their nature for \( y = x^7 + 3x^6 + 3x^5 + x^4 \).

Solution: \( y' = 7x^6 + 18x^5 + 15x^4 + 4x^3 = x^3(7x^3 + 18x^2 + 15x + 4) = 0 \). Clearly \( x = 0 \) is one solution, but it’s a lot of hard work to find the others. We’ll skip over the method of factorising
\[7x^3 + 18x^2 + 15x + 4\] so that we can illustrate how the First and Second Derivative Tests manage with the job of determining the nature of the stationary points.

With a certain amount of hard work we can factorize \(\frac{dy}{dx}\) factorises as \(\frac{dy}{dx} = x^3(x + 1)^2(7x + 4)\).

So there are three stationary points, at \(x = 0\), \(-1\) and \(-\frac{4}{7}\).

Let’s use the Second Derivative Test.
\[y'' = 42x^5 + 90x^4 + 60x^3 + 12x^2.\]
When \(x = 0\), \(y'' = 0\) so the test fails.
When \(x = -1\), \(y'' = 0\) so the test fails again.

When \(x = -\frac{4}{7}\), \(y'' = 42\left(-\frac{4}{7}\right)^5 + 90\left(-\frac{4}{7}\right)^4 + 60\left(-\frac{4}{7}\right)^3 + 12\left(-\frac{4}{7}\right)^2\). Oh, forget it! It’s too tedious to work out. So the Second Derivative Test fails dismally. In two out of the three cases it can’t give us an answer at all. It could handle the third case but the arithmetic is pretty horrible. Let’s see whether the First Derivative Test manages better.

\[y' = 7x^6 + 18x^5 + 15x^4 + 4x^3 = x^3(x + 1)(7x^2 + 11x + 4) = x^3(x + 1)^2(7x + 4).\]

When \(x = 0\) then \(y' = --+ = 0\).
When \(x = 0^+\) then \(y' = +++ = +\)
There is a local minimum at \(x = 0\).

When \(x = -1^-\) then \(y' = -- = +\)
When \(x = -1^+\) then \(y' = + - = +\)
There is a stationary point of inflection at \(x = -1\).

When \(x = -(4/7)^-\) then \(y' = -- = +\)
When \(x = -(4/7)^+\) then \(y' = + + = -\)
There is a local maximum at \(x = -(4/7)\).

In this example the First Derivative Test worked smoothly while the Second Derivative Test fell over rather badly. Now in all fairness it should be pointed out that this example was specially chosen so as to show the Second Derivative Test at its worst. It’s not always as bad as that and you should always think of using it as your first option. But be ready to jump ship in favour of the First Derivative Test.

\[\text{The Second Derivative Test is often simple and effective BUT}\]
\[\bullet\] it’s sometimes inconclusive
\[\bullet\] it can involve much unnecessary calculation
\[\bullet\] and it disguises what’s going on.
\[\text{Be ready to switch to the First Derivative Test should you need to.}\]

\[\text{§5.5 Global Maximum and Minimum}\]

Local maxima and minima are all very interesting. But more usually we want to find a \emph{global} maximum or minimum. This is the overall largest or smallest value of \(y\) over an entire range of values.

Suppose a manufacturer discovers that if he varies the price of an item a little bit up or down from $3 he’ll make less profit than if he charges $3 exactly. He might conclude that $3 represents a local maximum for profit, and that’s a reasonable conclusion. So how much should he charge? Three dollars? Not necessarily. It might be that if he increased the price enough then sales revenue
might start to increase again. Sometimes people avoid buying something if it appears too cheap. It might well be that if he priced the item at $9.99 he’ll make more profit than if he charged any other price. This, then, should be the price he should charge.

Sometimes you have to go down from a maximum in order to go up even higher. Every mountain climber knows that if you want to scale the highest peak you’ll have to spend a certain amount of time going up and down smaller peaks.

**The GLOBAL MAXIMUM (MINIMUM)**

_of \( f(x) \) for \( x \) in a certain range of values, is the LARGEST (SMALLEST) value of \( f(x) \) for \( x \) in that range._

A local maximum need not be a global one. The curve could go even higher at another stage. A global maximum is very often one of the local maxima, but not always. It could be an endpoint of the range of values.

In this example the global minimum is also a local one. But the global maximum occurs at the right-hand end-point.

**Example 8:** Find the global maximum and minimum of \( f(x) = x^3 - 3x^2 - 9x + 30 \) over the interval \([0, 5]\)

**Solution:** \( f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x + 1)(x - 3). \)

So there are stationary points at \( x = -1 \) and \( x = 3 \) but only the one at \( x = 3 \) is relevant to the interval \([0, 5]\). The end-points are \( x = 0 \) and \( x = 5 \).

The global maximum and the global minimum, over this interval, will occur at one of the three values: \( 0, 3, 5 \). Now \( f(0) = 30, f(3) = 3 \) and \( f(5) = 35 \). Clearly the global maximum is 35 (occurring at \( x = -1 \)) and the global minimum is 3, at \( x = 3 \).

If the range of values is \([−2, 4]\) then both stationary points would be relevant.

Since \( f(-2) = 28, f(-1) = 35, f(3) = 3, f(4) = 10 \) the global maximum is 35 (at \( x = -1 \)) and the global minimum is 3 (at \( x = 3 \)).
If the range of values is \([0, \infty)\), that is all \(x \geq 0\), the global minimum would be 3 (at \(x = 3\)), but there would be no global maximum.

It’s clear that there can’t ever be more than one global maximum, though this maximum can occur for several different values of \(x\). But there needn’t be one at all and you must remember to be aware of this possibility.

There might be no global maximum if the range of values excludes an endpoint. This might be because an interval is infinite. For example there’s a global minimum for the function \(f(x) = x^2\) over the whole real line but no global maximum.

There might be no global maximum or minimum on a finite interval if one or both endpoints are excluded. For example there’s no global maximum to \(f(x) = 1 - x\) over the interval \((0, 1]\), that is for \(0 < x \leq 1\). You can get values of \(f(x)\) as close as you want to 1 but because \(x = 0\) is excluded from the interval, you can never exactly reach the value 1 for \(x\) in this range.

Another case where you might have no local maximum or minimum is where there’s a discontinuity in the function. For example \(f(x) = 1/x\) over the range \([-3, 3]\) has no global maximum and no global minimum. This is because, by taking values of \(x\) close enough to \(x = 0\) we can get positive or negative values of \(f(x)\) with magnitudes (absolute values) as large as we like.

How to Locate the Global Maximum and Global Minimum (where they exist).

1. Check whether a global maximum or minimum exists.
2. Find the stationary points.
3. Find the end-points.
4. Find any points where the derivative doesn’t exist.
5. Evaluate the function at each of these points.
6. Select the smallest and largest of these values.

§5.6 Partial Derivatives

An equation of the form \(y = f(x)\) represents a curve in the \(x-y\) plane. For every \(x\) we calculate \(y\) and plot the point \((x, y)\). If we were able to do this for every value of \(x\) these points would make up the curve. An equation of the form \(z = f(x, y)\) represents a surface in 3-dimensional space. Here we have 3 coordinate axes, the \(x\)-axis, the \(y\)-axis and the \(z\)-axis. Each of these axes is at right angles to the other two.
We consider the $z$-axis to be vertical and the $x$-axis and the $y$-axis, making up the $x$-$y$ plane, form a horizontal plane. The origin is where all three axes meet.

Every point in the 3-dimensional space has three coordinates $(x, y, z)$. A point $(x, y, 0)$ lies in the $x$-$y$ plane at the place where we would plot $(x, y)$ if we were only working in two dimensions only. The point $(x, y, z)$ lies $z$ units directly above $(x, y, 0)$. If $z$ is negative we interpret this as being below.

An equation of the form $z = f(x, y)$ then, represents a surface in 3-dimensional space. It’s hard to actually draw this surface because they don’t make 3-dimensional graph paper, but you can imagine the act of drawing such a surface. For each combination of $x, y$ we calculate $z = f(x, y)$ and plot the point $(x, y, z)$ in the air. If we do this for all combinations of $x$ and $y$ (or at least all $x, y$ for which the function exists) these points will constitute the surface.

**Example 9:** Describe the surface $z = 1 - x^2 - y^2$.

**Solution:** Squaring we get $z^2 = 1 - x^2 - y^2$, which can be written as $x^2 + y^2 + z^2 = 1$.

Now the distance of the point $(x, y, z)$ from the origin is $\sqrt{x^2 + y^2 + z^2}$.

To see this, let $P$ be the point $(x, y, z)$ and let $Q$ be the foot of the perpendicular to the $x$-$y$ plane. In other words $Q$ is the point $(x, y, 0)$, the point in the $x$-$y$ plane that lies directly below (or above) $P$. Let $R$ be the point $(x, 0, 0)$.

Now $\triangle PQR$ is a right-angled triangle (the right angle is at $Q$) and so is $\triangle OPR$ (right-angled at $R$). They don’t look like right angles on the diagram because this is a perspective diagram of something 3-dimensional.

By Pythagoras’ Theorem in $\triangle PQR$, $PR^2 = y^2 + z^2$. $PR$ is the hypotenuse in that triangle but in $\triangle OPR$ it is just one of the other two sides. The other side is $OR$ and the hypotenuse is now $OP$. So $OP^2 = PR^2 + x^2 = x^2 + y^2 + z^2 + x^2$. Taking (positive) square roots we therefore get $OP = \sqrt{x^2 + y^2 + z^2}$.

For this surface, $x^2 + y^2 + z^2 = 1$ the points are all at a distance of 1 unit from the origin and so they all lie on the sphere of radius 1 with its centre at the origin.

But the surface is not the whole sphere. Since $z = \sqrt{1 - x^2 - y^2}$, it’s the positive (or zero) square root of $1 - x^2 - y^2$. Remember that positive numbers have two square roots but the square root sign, $\sqrt{\cdot}$, denotes the positive one. So on this surface $z \geq 0$ so all points lie on or above the $x$-$y$ plane. The surface is therefore a hemisphere.
Slope in 3 dimensions is a little harder to define. As before, we take a tangent at a point, but in 3-dimensions there are infinitely many such tangents. There’s a plane which touches the surface at the given point and any line in this plane will be a tangent. The problem is that they usually have different slopes. So instead of finding the slope at a point we have to content ourselves with finding the slope in different directions.

When you’re out walking on the side of a bald hill, one where you’re free to walk in any direction, you’ll have noticed that some directions are steeper than others. You can walk directly up towards the summit, but if that’s too steep you may choose to walk in a different direction where the slope is less. You’ll have to walk further to get to the top, but it might be less tiring. Road engineers won’t build a road directly up a hill, if it’s too steep, but will wind the road around, gradually climbing but less steeply. And if you have no desire to get to the top of the hill you can always walk around it along a level path, where the slope is zero.

So for each direction in the \(x-y\) plane there’s a slope for the surface. But there are two special directions that we select – the directions of the \(x\)- and \(y\)-axes. If we move in the direction of the \(x\)-axis on the surface we’re keeping the \(y\)-values fixed. This slope is called the \(x\)-derivative and it is denoted by \(\frac{\partial z}{\partial x}\). Here we use a different sort of “d”, the lower case Greek letter “d”. We write it differently to \(\frac{dy}{dx}\) in order to remind us that we’re only varying \(x\). All other independent variables (in this case that’s just \(y\)) are kept fixed.

If instead we move in the \(y\)-direction, keeping \(x\) fixed, we get the \(y\)-derivative, denoted by \(\frac{\partial z}{\partial y}\). There are other directions we could have used, but it’s possible to work out slopes in other directions from just these two.

These derivatives, \(\frac{\partial z}{\partial x}\) and \(\frac{\partial z}{\partial y}\), are called **partial derivatives**.

<table>
<thead>
<tr>
<th>PARTIAL DERIVATIVES</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{\partial z}{\partial x}) is the derivative of (z) with respect to (x)</td>
</tr>
<tr>
<td>keeping (y) fixed</td>
</tr>
<tr>
<td>(\frac{\partial z}{\partial y}) is the derivative of (z) with respect to (y)</td>
</tr>
<tr>
<td>keeping (x) fixed</td>
</tr>
</tbody>
</table>

Working out partial derivatives is no more difficult than ordinary differentiation. All you have to do is to remember which variable is being kept fixed and to treat it as if it was constant.

**Example 10:** If \(z = x^2 - y^2 + 3x + y + 3xy\), find \(\frac{\partial z}{\partial x}\) and \(\frac{\partial z}{\partial y}\).

**Solution:** \(\frac{\partial z}{\partial x} = 2x - 0 + 3 + 0 + 3y = 2x + 3y + 3\). The \(x\)-derivative of \(y^2\) and \(y\) are both zero because \(y\) is being treated as a constant. The \(x\)-derivative of \(3xy\) is \(3y\) because \(3xy = (3y)x\) and the \(3y\) is the constant in front of the \(x\).

Similarly, \(\frac{\partial z}{\partial y} = -2y + 1 + 3x\).
§5.7 Maxima and Minima for Functions of Two Variables

At the top of a hill it doesn’t matter in which direction you walk – it’s flat in every direction, for an instant. The tangent plane is horizontal and so all the tangents have zero slope. The same holds at a local minimum.

So to find local maxima and minima for a function of two variables \( z = f(x, y) \) you look for points where both \( \frac{\partial z}{\partial x} = 0 \) and \( \frac{\partial z}{\partial y} = 0 \). These points are called stationary points.

A stationary point for a function \( z = f(x, y) \)
is a point where \( \frac{\partial z}{\partial x} = 0 \) and \( \frac{\partial z}{\partial y} = 0 \).

Example 11: Find the stationary points of \( z = x^2 - xy + y^2 - 3y \).

Solution: \( \frac{\partial z}{\partial x} = 2x - y \) and \( \frac{\partial z}{\partial y} = -x + 2y - 3 \).

We therefore have to solve the following system of simultaneous equations.

\[
\begin{align*}
2x - y &= 0 \\
-x + 2y &= 3
\end{align*}
\]

Multiplying the first equation by 2 and adding the second equation (in order to eliminate \( y \)) we get: 
\( 3x = 3, \) or \( x = 1. \)

Substituting into the first equation (either equation can be used) we get \( y = 2. \)

When \( x = 1 \) and \( y = 2, z = -3. \) So there’s one stationary point at \( (1, 2, -3) \).

The determination of the nature of stationary points is considerably more complicated than in the one variable case. As well as stationary points of inflection there are stationary points called “saddle points”.

If you think of a saddle that one puts onto a horse, it slopes down on either side in the left and right directions, following the contour of the horse’s back. But it slopes up at the front and the back. There’s a stationary point in the middle where you sit. It’s a local maximum in one direction but a local minimum in the other.

Your own body contains many saddle points. You can hold your arm, bent at right angles, so that on the inside of your elbow there’s a local minimum in the direction of your arm, but a local maximum at right angles to this. This is another example of a saddle point.

It’s possible to determine the nature of a stationary point for a function of two variables using second derivatives. But the situation is rather more complicated because there are not only the second derivatives in the two directions \( \frac{\partial^2 z}{\partial x^2} \) and \( \frac{\partial^2 z}{\partial y^2} \) but also mixed second derivatives where you differentiate once with respect to one of the variables, holding the other one constant, and then differentiating with respect to the variable that was held fixed the first time. You’ll be pleased to learn that we won’t get embroiled in these complications here.
EXERCISES FOR CHAPTER 5

Exercise 1: Find the stationary points of the following functions and determine their nature (there is no need to compute the corresponding y-values):

(i) \( y = x^2 - 8x + 17 \);
(ii) \( y = 2x^3 - 3x^2 - 12x + 10 \);
(iii) \( y = x^3 - 9x^2 + 27x - 26 \);
(iv) \( y = (x + 1)^6 \);
(v) \( y = \sqrt{x + 1} \);
(vi) \( y = x - \sqrt{x} \);
(vii) \( y = (x - 1)^5 \);
(viii) \( y = x\sqrt{x + 1} \);
(ix) \( y = x\sqrt{18 - x^2} \);
(x) \( y = (x - 1)^2(x + 1)^3 \);
(xi) \( y = 1 - x + 2\sqrt{x + 1} \).

Exercise 2: Find the global maximum and minimum of the following functions (these are the same as above) over the stated interval:

(i) \( y = x^2 - 8x + 17 \) over \([0, 5]\);
(ii) \( y = 2x^3 - 3x^2 - 12x + 10 \) over \([-2, 1]\);
(iii) \( y = x^3 - 9x^2 + 27x - 26 \) over \([2, 4]\);
(iv) \( y = (x + 1)^6 \) over \([-2, 0]\);
(v) \( y = \sqrt{x + 1} \) over \([0, 3]\);
(vi) \( y = x - \sqrt{x} \) over \([0, 1]\);
(vii) \( y = x(x - 1)^5 \) over \([1, 2]\);
(viii) \( y = x\sqrt{x + 1} \) over \([-1, 3]\);
(ix) \( y = x\sqrt{18 - x^2} \) over \([-4, 4]\);
(x) \( y = (x - 1)^2(x + 1)^3 \) over \([0, 1]\);
(xi) \( y = 1 - x + 2\sqrt{x + 1} \) over \([0, 3]\).

Exercise 3: For each of the following functions of two variables find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \):

(i) \( z = x + 2y \);
(ii) \( z = xy \);
(iii) \( z = x^3y^3 \);
(iv) \( z = x^2 + 3y^2 - xy + x + 5y + 7 \);
(v) \( z = 6x^2 - 5xy + 4y^2 - 3x + 2y - 1 \).

Exercise 4: Each of the following equations describe a surface with one stationary point. Find it, in each case. (You don’t need to find the corresponding z-coordinate. Nor do you need determine its nature.)

(i) \( z = x^2 + y^2 \);
(ii) \( z = x^2 + 2y^2 - 3x - 4y + 5 \);
(iii) \( z = x^2 + xy + y \);
(iv) \( z = x^2 - y^2 + 2xy + 2x - 6y + 1 \);
(v) \( z = x^2 + 3y^2 - xy + x + 5y + 7 \);
(vi) \( z = 5x^2 + 3y^2 - 4xy - 8x - 10y + 6 \).

Exercise 5: Find the stationary points of the surface \( z = x^2y - 4x - 4y \).
Exercise 1:

(i) \( \frac{dy}{dx} = 2x - 8 = 0 \) when \( x = 4 \).

**Second Derivative Test:** \( \frac{d^2y}{dx^2} = 2 > 0 \) so \( 4, 1 \) is a local minimum.

**First Derivative Test:** \( \frac{dy}{dx} = 2(x - 4) \).

When \( x = 4^- \), \( \frac{dy}{dx} = - \)

When \( x = 4^+ \), \( \frac{dy}{dx} = + \)

So there’s a local minimum at \( x = 4 \).

[The Second Derivative Test is easier in this case.]

(ii) \( \frac{dy}{dx} = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x - 2)(x + 1) = 0 \) when \( x = 2 \) or \( -1 \).

**Second Derivative Test:** \( \frac{d^2y}{dx^2} = 12x - 6 \)

When \( x = 2 \), \( \frac{d^2y}{dx^2} > 0 \) so there’s a local minimum at \( x = 2 \).

When \( x = -1 \), \( \frac{d^2y}{dx^2} < 0 \) so there’s a local maximum at \( x = -1 \).

**First Derivative Test:** \( \frac{dy}{dx} = 6(x - 2)(x + 1) \).

When \( x = 2^- \), \( \frac{dy}{dx} = + + + - - \)

When \( x = 2^+ \), \( \frac{dy}{dx} = + + + + + \)

So there’s a local minimum at \( x = 2 \).

When \( x = -1^- \), \( \frac{dy}{dx} = + - - - + \)

When \( x = -1^+ \), \( \frac{dy}{dx} = + - - + - \)

So there’s a local maximum at \( x = -1 \).

[Again, the Second Derivative Test is easier in this case.]

(iii) \( \frac{dy}{dx} = 3x^2 - 18x + 27 = 3(x^2 - 6x + 9) = 3(x - 3)^2 = 0 \) when \( x = 3 \).

**Second Derivative Test:** \( \frac{d^2y}{dx^2} = 6x - 18 \)

When \( x = 3 \), \( \frac{d^2y}{dx^2} = 0 \) so the SECOND DERIVATIVE TEST FAILS.

**First Derivative Test:** \( \frac{dy}{dx} = 3(x - 3)^2 \).

When \( x = 3^- \), \( \frac{dy}{dx} = + \)

When \( x = 3^+ \), \( \frac{dy}{dx} = + \)

So there’s a stationary point of inflection at \( x = 3 \).
[Here we had no choice – we had to use the First Derivative Test.]

**(iv)** \( \frac{dy}{dx} = 6(x + 1)^5 = 0 \) when \( x = -1 \).

**Second Derivative Test:** \( \frac{d^2y}{dx^2} = 30(x + 1)^4 \).

When \( x = -1 \), \( \frac{d^2y}{dx^2} = 0 \) so the Second Derivative Test Fails.

**First Derivative Test:** \( \frac{dy}{dx} = 6(x + 1)^5 \).

When \( x = -1^− \), \( \frac{dy}{dx} = -

When \( x = -1^+ \), \( \frac{dy}{dx} = +

So there’s a local minimum at \( x = -1 \).

[Again, we had no choice – we had to use the First Derivative Test.]

**(v)** \( \frac{dy}{dx} = \frac{1}{2\sqrt{x} + 1} \) > 0 for all \( x \), so this function doesn’t have any stationary points.

**(vi)** \( \frac{dy}{dx} = 1 - \frac{1}{2\sqrt{x}} = \frac{2\sqrt{x} - 1}{2\sqrt{x}} = 0 \) when \( 2\sqrt{x} = 1 \), ie \( \sqrt{x} = \frac{1}{2} \), ie \( x = \frac{1}{4} \).

**Second Derivative Test:** \( \frac{dy}{dx} = 1 - \frac{1}{2\sqrt{x}} \) so \( \frac{d^2y}{dx^2} = \frac{1}{4} x^{-3/2} = \frac{1}{4x\sqrt{x}} \).

When \( x = \frac{1}{4} \), \( \frac{d^2y}{dx^2} > 0 \) so there’s a local minimum at \( x = \frac{1}{4} \).

**First Derivative Test:** \( \frac{dy}{dx} = \frac{2\sqrt{x} - 1}{2\sqrt{x}} \).

When \( x = \frac{1}{4}^− \), \( \frac{dy}{dx} = \frac{-}{+} = -

When \( x = \frac{1}{4}^+ \), \( \frac{dy}{dx} = \frac{+}{+} = + \quad \frac{1}{4}

So there’s a local minimum at \( x = \frac{1}{4} \).

[Both tests appear to be equally good here.]

**(vii)** \( \frac{dy}{dx} = (x - 1)^5 + 5x(x - 1)^4 = (x - 1)^4(x - 1 + 5x) = (x - 1)^4(6x - 1) = 0 \) when \( x = 1, \ \frac{1}{6} \).

**Second Derivative Test:** \( \frac{d^2y}{dx^2} = 4(x - 1)^3(6x - 1) + 6(x - 1)^4 \\
= 2(x - 1)^3(12x - 2 + 3x - 3) \\
= 10(x - 1)^3(3x - 1). \)

When \( x = 1 \), \( \frac{d^2y}{dx^2} = 0 \) so the Second DERIVATIVE TEST FAILS.

When \( x = \frac{1}{6} \), \( \frac{d^2y}{dx^2} > 0 \) so there’s a local minimum at \( x = \frac{1}{6} \).
First Derivative Test: \( \frac{dy}{dx} = (x - 1)^4 (6x - 1) \).

When \( x = \frac{1}{6} \), \( \frac{dy}{dx} = - - - - \)

When \( x = \frac{1}{6} + \), \( \frac{dy}{dx} = + + + + \)

So there’s a local minimum at \( x = \frac{1}{6} \).

When \( x = 1^- \), \( \frac{dy}{dx} = + + = + \)

When \( x = 1^+ \), \( \frac{dy}{dx} = + + = + \)

So there’s a stationary point of inflection at \( x = 1 \).

[Whether we like it or not we have to use the First Derivative Test here.]

(viii) \( \frac{dy}{dx} = \sqrt{x + 1} + \frac{x}{2\sqrt{x + 1}} = \frac{2(x + 1) + x}{2\sqrt{x + 1}} = \frac{3x + 1}{2\sqrt{x + 1}} = 0 \) when \( x = -\frac{1}{3} \).

Second Derivative Test: \( \frac{d^2y}{dx^2} = \sqrt{x + 1} + \frac{x}{2} (x + 1)^{-1/2} \).

So \( \frac{d^2y}{dx^2} = \frac{1}{2\sqrt{x + 1}} \times \frac{1}{2} (x + 1)^{-1/2} + \frac{x}{2} (-\frac{1}{2})(x + 1)^{-3/2} \)

\[ = \frac{(x + 1)^{-3/2}}{4} \times [2(x + 1) + 2(x + 1) - x] = \frac{(x + 1)^{-3/2}}{4} (3x + 4). \]

When \( x = -\frac{1}{3} \), \( \frac{d^2y}{dx^2} > 0 \) so there’s a local minimum at \( x = -\frac{1}{3} \).

First Derivative Test: \( \frac{dy}{dx} = \frac{3x + 1}{2\sqrt{x + 1}} \).

When \( x = -\frac{1}{3} - \), \( \frac{dy}{dx} = - = - \)

When \( x = -\frac{1}{3} + \), \( \frac{dy}{dx} = + = + \)

So there’s a local minimum at \( x = -\frac{1}{3} \).

[While the Second Derivative Test works it involves a messy second differentiation. The First Derivative Test is much easier here. MORAL: If it looks as though you’ll have a hard time differentiating a second time, abandon the Second Derivative Test and go for the First Derivative Test.]

(ix) \( \frac{dy}{dx} = \sqrt{18 - x^2} + x \cdot \frac{1}{2\sqrt{18 - x^2}} (-2x) = \frac{2(18 - x^2) - 2x^2}{2\sqrt{18 - x^2}} = \frac{36 - 4x^2}{2\sqrt{18 - x^2}} = 0 \) when \( x = \pm 3 \).

Second Derivative Test: is too tedious.

First Derivative Test: \( \frac{dy}{dx} = \frac{36 - 4x^2}{2\sqrt{18 - x^2}} \).

When \( x = 3^- \), \( \frac{dy}{dx} = + = + \)

When \( x = 3^+ \), \( \frac{dy}{dx} = - = - \)

So there’s a local maximum at \( x = 3 \).
When \( x = -3^- \), \( \frac{dy}{dx} = - + = - \\
When \( x = -3^+ \), \( \frac{dy}{dx} = + + = + \\
So there’s a local minimum at \( x = -3 \).

\[ (x) \frac{dy}{dx} = (x - 1)(x + 1)^2 (5x - 1) \]

First Derivative Test: \( \frac{dy}{dx} = (x - 1)(x + 1)^2 (5x - 1) \).

When \( x = 1^- \), \( \frac{dy}{dx} = + = + = + \\
When \( x = 1^+ \), \( \frac{dy}{dx} = + = + = + \\
So there’s a local minimum at \( x = 1 \).

When \( x = -1^- \), \( \frac{dy}{dx} = + = + = + \\
When \( x = -1^+ \), \( \frac{dy}{dx} = + = + = + \\
So there’s a stationary point of inflection at \( x = -1 \).

When \( x = \frac{1}{5}^- \), \( \frac{dy}{dx} = + = + = + \\
When \( x = \frac{1}{5}^+ \), \( \frac{dy}{dx} = + = + = + \\
So there’s a local maximum at \( x = \frac{1}{5} \).

Exercise 2:
(i) The stationary point is at \( x = 4 \). The end-points are at \( x = 0 \) and \( x = 5 \).

| \( x \) | 0 | 4 | 5 |
| \( y \) | 17 | 1 | 2 |

So the global maximum is 17 (at \( x = 0 \)) and the global minimum is 1 (at \( x = 4 \)).

(ii) The stationary points are at \( x = -1 \) and at \( x = 2 \), but \( x = 2 \) lies outside the interval.

The end-points are at \( x = -2 \) and \( x = 1 \).

| \( x \) | -2 | -1 | 1 |
| \( y \) | 6 | 17 | -3 |
So the global maximum is 17 (at $x = -1$) and the global minimum is $-3$ (at $x = 1$).

(iii) There are no local maxima and minima. The end-points are at $x = 2$ and $x = 4$.

\[
\begin{array}{c|c|c}
 x & 2 & 4 \\
 y & 0 & 2 \\
\end{array}
\]

So the global maximum is 2 (at $x = 4$) and the global minimum is 0 (at $x = 4$).

(iv) The stationary points is at $x = -1$. The end-points are at $x = -2$ and $x = 0$.

\[
\begin{array}{c|c|c}
 x & -2 & -1 \\
 y & 1 & 0 \\
\end{array}
\]

So the global maximum is 1 (at $x = -2$ and also at $x = 0$) and the global minimum is 0 (at $x = -1$).

(v) There are no stationary points. The end-points are at $x = 0$ and $x = 3$.

\[
\begin{array}{c|c|c}
 x & 0 & 3 \\
 y & 1 & 2 \\
\end{array}
\]

So the global maximum is 2 (at $x = 3$) and the global minimum is 1 (at $x = 0$).

(vi) The stationary point is at $x = 1/4$. The end-points are at $x = 0$ and $x = 1$.

\[
\begin{array}{c|c|c}
 x & 0 & 1/4 \\
 y & 0 & -1/4 \\
\end{array}
\]

So the global maximum is 0 (at $x = 0$ and at $x = 1$) and the global minimum is $-1/4$ (at $x = 1/4$).

(vii) The only stationary point is at $x = 1/6$ but this is outside the interval. The end-points are at $x = 1$ and $x = 2$.

\[
\begin{array}{c|c|c}
 x & 1 & 2 \\
 y & 0 & 2 \\
\end{array}
\]

So the global maximum is 2 (at $x = 2$) and the global minimum is 0 (at $x = 1$).

(viii) The stationary point is at $x = -1/3$. The end-points are at $x = -1$ and $x = 3$.

\[
\begin{array}{c|c|c|c}
 x & -1 & -1/3 & 3 \\
 y & 3 & \sqrt{2}/3 & 6 \\
\end{array}
\]

So the global maximum is 6 (at $x = 3$) and the global minimum is $-\sqrt{2}/3\sqrt{3}$ (at $x = -1/3$).

(ix) The stationary points are at $x = -3$ and at $x = 3$. The end-points are at $x = -4$ and $x = 4$.

\[
\begin{array}{c|c|c|c}
 x & -4 & -3 & 3 & 4 \\
 y & -4\sqrt{2} & -9 & 9 & 4\sqrt{2} \\
\end{array}
\]

So the global maximum is 9 (at $x = -3$) and the global minimum is $-9$ (at $x = -3$).

(x) The stationary points are at $x = -1$, at $x = 1/5$ and at $x = 1$, but $x = -1$ lies outside the interval. The end-points are at $x = 0$ and $x = 1$.

\[
\begin{array}{c|c|c}
 x & 0 & 1/5 & 1 \\
 y & 1 & -3456/3125 & 0 \\
\end{array}
\]
So the global maximum is at approximately 1.10 (at $x = 0.2$) and the global minimum is 0 (at $x = 1$).

(xi) The stationary point is at $x = 0$. The end-points are at $x = 0$ and $x = 3$.

So the global maximum is 3 (at $x = 0$) and the global minimum is 2 (at $x = 3$).

Exercise 3:

(i) $\frac{\partial z}{\partial x} = 2x - y + 1; \quad \frac{\partial z}{\partial y} = 6y - x + 5$.

(ii) $\frac{\partial z}{\partial x} = y; \quad \frac{\partial z}{\partial y} = x$.

(iii) $\frac{\partial z}{\partial x} = 2xy^3; \quad \frac{\partial z}{\partial y} = 3x^2y^3$.

(iv) $\frac{\partial z}{\partial x} = 2x - y + 1; \quad \frac{\partial z}{\partial y} = 3y - x + 5$.

(v) $\frac{\partial z}{\partial x} = 12x - 5y - 3; \quad \frac{\partial z}{\partial y} = -5x + 8y + 2$.

Exercise 4:

(i) $\frac{\partial z}{\partial x} = 2x = 0$ and $\frac{\partial z}{\partial y} = 2y = 0$ so $x = y = 0$. The only stationary point is (0, 0).

(ii) $\frac{\partial z}{\partial x} = 2x - 3 = 0$ and $\frac{\partial z}{\partial y} = 4y - 4 = 0$ so $x = \frac{3}{2}, y = 1$. The only stationary point is $(3/2, 1)$.

(iii) $\frac{\partial z}{\partial x} = 2x + y = 0$ and $\frac{\partial z}{\partial y} = x + 1 = 0$ so $x = -1$ and $y = -2x = 2$. The stationary point is $(-1, 2)$.

(iv) $\frac{\partial z}{\partial x} = 2x + 2y + 2 = 0$ and $\frac{\partial z}{\partial y} = -2y + 2x - 6 = 0$ so

\[
2x + 2y = -2 \quad \text{and} \quad 2x - 2y = 6.
\]

Adding, we get $4x = 4$, so $x = 1$.

Substituting into either of the two equations gives $y = -2$. The stationary point is $(1, -2)$.

(v) $\frac{\partial z}{\partial x} = 2x - y + 1 = 0$ and $\frac{\partial z}{\partial y} = 6y - x + 5 = 0$.

We must now solve the following equations simultaneously:

\[
2x - y + 1 = 0 \quad \text{and} \quad 6y - x + 5 = 0.
\]

It helps if we rearrange terms so that the x’s and y’s are underneath one another:

\[
2x - y + 1 = 0 \quad \text{and} \quad -x + 6y + 5 = 0.
\]

Let’s choose to eliminate $x$. So multiply the second equation by 2 giving the two equations:

\[
2x - y + 1 = 0 \quad \text{and} \quad -2x + 12y + 10 = 0.
\]

We can now add these equations to make the x’s disappear: $11y + 11 = 0$.

So $y = -1$. Substituting back into either of the equations gives $x = -1$.

The stationary point is $(-1, -1)$.

We could have eliminated $y$ instead. In this case we would have multiplied the first equation by 6:

\[
12x - 6y + 6 = 0 \quad \text{and} \quad -x + 6y + 5 = 0.
\]
We can now add these equations to make the y’s disappear: 11x + 11 = 0, giving x = -1. Substituting back into either of the equations gives y = -1. This leads to the same answer as before: a stationary point at (-1, -1).

\[(vi) \frac{\partial z}{\partial x} = 10x - 4y - 8 = 0 \text{ and } \frac{\partial z}{\partial y} = 6y - 4x - 10 = 0.\]

We must now solve the following equations simultaneously:

\[10x - 4y - 8 = 0 \text{ and } 6y - 4x - 10 = 0.\]

It helps if we rearrange terms so that the x’s and y’s are underneath one another:

\[10x - 4y - 8 = 0 \text{ and } -4x + 6y - 10 = 0.\]

Let’s choose to eliminate y. We multiply the first equation by 3 and the second by 2, giving:

\[30x - 12y - 24 = 0 \text{ and } -8x + 12y - 20 = 0.\]

We can now add the equations to eliminate y: 22x - 44 = 0, giving x = 2. Substituting back into either of the equations gives x = 3. The stationary point is (2, 3).

**Exercise 5:** \[\frac{\partial z}{\partial x} = 2xy - 4 = 0 \text{ and } \frac{\partial z}{\partial y} = x^2 - 4 = 0.\] So x = ±2 and xy = 2.

If x = 2, y = 1 and if x = -2, y = -1. So there are two stationary points: (2, 1) and (-2, -1).