2. PERCENTAGES

§2.1. Small Proportions

We know what we mean when we say that half of all pupils at a school are girls, or three quarters are from overseas. But when the amounts are small fractions are not the best way of describing proportions. We know what we mean when we say that one in every 20 men are unemployed, but if it is the case that one in every 17.6 women are unemployed that seems a bit clumsy.

A better system is to use percentages. The word “per cent” is really made up of two Latin words: “per” meaning “for every” and “cent” meaning “a hundred”. So instead of saying “one in every 20” we would say “five in every hundred” or “five percent”. And “instead of saying “one in every 17.6” we would say “5.68 in every hundred” or “5.68 percent”. Where did the 5.68 come from? Well we divided 100 by 17.6 and got approximately 5.68.

There is a special symbol for percentages. It is “%”. So “five percent would be written as 5%. Just remember that this means 5 in every hundred. And 5.68% is easier to grasp than one in 17.6. It’s interesting that the % symbol is made up of a line and two circles. If you rearrange these you can get \(\frac{100}{\text{\textcircled{}}}\), or 100. When you see the percentage sign you can see the number 100.

To convert a fraction into a percentage we simply multiply by 100. So “one in 20” as a fraction is \(\frac{1}{20}\) and so as a percentage it is \(\frac{1}{20} \times 100\) percent or 5%. One in 17.6 is \(\frac{1}{17.6}\) and so as a percentage it is \(\frac{1}{17.6} \times 100 = 5.68\)%. Sometimes we have fractions that are not just one in so many. We might say that three quarters of families have more than one child. As a fraction, three quarters is \(\frac{3}{4}\). As a percentage it is \(\frac{3}{4} \times 100\) per cent, which is 75%.

TO CONVERT A FRACTION TO A PERCENTAGE MULTIPLY BY 100.

Example 1: What is \(\frac{5}{8}\) as a percentage?

Solution: Multiply by 100. So it is \(\frac{5}{8} \times 100\) percent, that is 62.5%. Note that \(\frac{500}{8} = \frac{125}{2} = 62.5\).

A special case is where the fraction is the whole lot. We might say “all dentists recommend Smilodent toothpaste”. As a fraction this is just 1. For example if there were 20 dentists in the survey, and they all recommended Smilodent, the fraction would be \(\frac{20}{20} = 1\). As a percentage this would be 100%. So 100% of dentists recommend Smilodent.
There are certain other fractions that occur frequently and where, with bit of practice, you remember them and don’t have to work them out. Check one or two of them in this list to assure yourself that in each case we have simply multiplied by 100.

<table>
<thead>
<tr>
<th>FRACTION</th>
<th>PERCENTAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>50%</td>
</tr>
<tr>
<td>1/3</td>
<td>33% (actually 33.33…%)</td>
</tr>
<tr>
<td>2/3</td>
<td>67% (actually 66.66…%)</td>
</tr>
<tr>
<td>1/4</td>
<td>25%</td>
</tr>
<tr>
<td>3/4</td>
<td>75%</td>
</tr>
<tr>
<td>1/5</td>
<td>20%</td>
</tr>
<tr>
<td>2/5</td>
<td>40%</td>
</tr>
<tr>
<td>3/5</td>
<td>60%</td>
</tr>
<tr>
<td>4/5</td>
<td>80%</td>
</tr>
<tr>
<td>1/10</td>
<td>10%</td>
</tr>
<tr>
<td>1/20</td>
<td>5%</td>
</tr>
<tr>
<td>1/100</td>
<td>1%</td>
</tr>
</tbody>
</table>

To convert a percentage back to a fraction we simply divide by 100. So 35% is a fraction of $\frac{35}{100} = \frac{7}{20}$.

**TO CONVERT A PERCENTAGE TO A FRACTION DIVIDE BY 100.**

**Example 2:** What is 12% as a fraction?

**Solution:** Just remember that this just means 12 in every 100, so 12% is $\frac{12}{100} = \frac{3}{25}$.

**§2.2. Small Changes**

Frequently a percentage describes a change in something. For example if you invest money you get interest, and the interest is proportional to the amount of money you have invested. If you invest $100 at 3% interest, the amount of interest after one year is $3. Interest rates are usually quoted “per annum”, which is Latin for “every year”.

But the important thing is that, at the end of a year, you now have $103. It is important to remember that in the second year you get 3% of $103, that is $\frac{3}{100} \times 103$ dollars,
that is $3.09. You get more interest in the second year because the interest from the first year also earns interest in the second. This is what is called **compound interest**. So after two years your balance will be $106.09.

Now one of the things that you quickly learn about percentages is that you can’t add or subtract them. You might think that 3% interest in the first year plus 3% interest in the second means that you earn 6% interest over the two years. But instead of having $106 you will have $106.09. So \(3\% + 3\% = 6.09\%\).

You see percentages don’t obey the usual rules of addition. If we left our $100 to accumulate over 20 years would we get 60%, that is $60 interest? No. Because of compound interest we would get over $80 after 20 years. Consider the following example.

**Example 3:** Suppose you invest $1000 in shares in a mining company. The first year you make 30% and the second year you make 20%. But in the third year you lose 50%. No worries. You are back to $1000 because 30% + 20% − 50% = 0%. Right?

Wrong!. After the first year your investment would be worth $1300. In the second year you earn 20% of $1300, which is $260. Your investment is now worth $1560. In the third year you suffer a loss of 50%. That’s half the value wiped off. Your investment is now worth only half of $1560, which is $780. This is a loss of $220, or 22%.

So, \(30\% + 20\% − 50\% = −22\%.\) This is a strange arithmetic indeed.

The moral of this example is:

**YOU CAN NEVER ADD OR SUBTRACT PERCENTAGES.**

There are two things you can do. You can work out the increases or decreases in dollars, or people, or whatever the items are. This is what we did in Example 3. Or you can **multiply factors**. This is by far the easiest method.

Every percentage increase or decrease is equivalent to multiplying by some factor. A 10% increase corresponds to a factor of 1.10. This is because $100 becomes $110. A 5% increase corresponds to a factor of 1.05. A 10% increase followed by a 5% increase corresponds to the factor 1.10 and 1.05, which we must **multiply**. This gives a combined factor of 1.155. Converting back to a percentage this is 15.5%.

A 20% decrease corresponds to a factor of 0.80. This is because $100 becomes $80 if we lose $20, or 20% and \(100 \times 0.80 = 80\). So a 20% increase followed by a 20% decrease corresponds to factors of 1.20 and 0.80. Since \(1.20 \times 0.80 = 0.96\) this represents a 4% decrease overall.

**TO CONVERT A PERCENTAGE INCREASE TO A FACTOR:**
**DIVIDE BY 100 AND ADD 1.**

**Example 5:** A 17% increase corresponds to a factor of 1.17. A 4% increase corresponds to a factor of 1.04. A 17\(\frac{1}{2}\)% increase corresponds to a factor of 1.175 (because \(17\frac{1}{2} = 17.5\)).

**TO CONVERT A PERCENTAGE DECREASE TO A FACTOR:**
**SUBTRACT FROM 100 AND THEN DIVIDE BY 100.**

**Example 6:** A 17% decrease corresponds to a factor of 0.83, since \(100 − 17 = 83\). This becomes 0.83 when we divide by 100. A 4% decrease corresponds to a factor of 0.96. A 17\(\frac{1}{2}\)% decrease corresponds to a factor of 0.825 (because \(100 − 17\frac{1}{2} = 82.5\)).
TO ADD OR SUBTRACT PERCENTAGES:

- CONVERT THE PERCENTAGES TO FACTORS
  - MULTIPLY THESE FACTORS
- CONVERT BACK TO A PERCENTAGE

Example 7: The population of a rare bird on an island is 800. In the following year there is a decrease of 15%. The next year there is a 10% increase and in the third year a 12% increase. In the fourth year there is a decrease of 7%. How many of these birds will there be at the end of the fourth year?

Solution: 15% decrease corresponds to a factor of 0.85. 10% increase corresponds to a factor of 1.10. 12% increase corresponds to a factor of 1.12. 7% increase corresponds to a factor of 0.93. Multiplying these factors we get $0.85 \times 1.1 \times 1.12 \times 0.93 = 0.973896$. This is a decrease of 2.6104% since $1 - 0.973896 = 0.026104$. But we don’t need to work out the percentage decrease. While we still have the overall decrease as a factor we simply multiply it by 800, to get $0.973896 \times 800 = 779.1168$. Of course we would report this as a population of 779 birds, or we might even round it off to 780.

But if we multiply factors when we add increases shouldn’t we divide when we multiply decreases? No, the different way we treat increases and decreases already takes into account as to whether they are increases or decreases. We always multiply the factors whatever the changes are.

Also you should note that the order of the increases and decreases doesn’t matter since we can multiply numbers in any order. This comes in handy if there are a lot of increases or decreases by the same amount.

Example 8: I invest $400 in the bank at 5% interest per annum. However at the end of year 3 the interest decreases to $3% but in the 10th year it goes back to 5%. How much will I have after 10 years?

Solution: The factor for 5% is 1.05 and for 3% it is 1.03. So the factor for the whole 10 years is $1.05 \times 1.05 \times 1.05 \times 1.03 \times 1.03 \times 1.03 \times 1.03 \times 1.03 \times 1.05$. This can be written as $1.05^4 \times 1.03^6$. Scientific calculators have a $y^x$ button. To get $1.05^4$ you enter $1.05 \ y^x \ 4$. For the whole calculation you would press the following buttons:

```
1 . 0 5 y^x 4 \times 1 . 0 3 y^x 6 =
```

The answer will be 1.451378…

Some calculators put in “^” when you press the $y^x$ button. The symbol “^” means “to the power of” and after putting in the power you must press the parenthesis button. Some calculators have an Ans button instead of an equals button.

We then multiply this by our initial investment of $400 to get $580.55, the amount that will be in our account after 10 years.