§1. Axiomatic Systems

As we have said, mathematical truth is established by logic, starting with some fundamental assumptions called axioms. One is obliged to accept the conclusions provided one accepts the logical principles used as well as the axioms. There is a real sense in which a set of axioms is a creed, like a religious creed.

Euclid is credited with devising the first set of axioms – the axioms for Geometry or, as we now consider it, the axioms for Euclidean Geometry. These axioms were considered to be “self-evident”. Axioms such as “between any two distinct points there is exactly one straight line”. Far from being self-evident, this is based on experimental evidence and has the same status as a scientific “fact”.

Axioms for other mathematical systems were proposed in the late 19th century. The first were the axioms of group theory. Never mind what group theory is or what the axioms are. Rather than self-evident truths they were considered to simply make up a definition of a group.

These days there is much controversy about gay marriage. Some regard it as self-evident that “marriage” means an arrangement between a man and a woman. In fact, it is merely the definition of the word “marriage”. Certainly there is no doubt that that is what was implied by the word over centuries. Others say the definition should be broadened. There’s a long history of the meaning of words being broadened. “Money” once referred to what we now call “currency” – coins and notes, but the meaning has been broadened to include electronic transactions. That does not mean that the meaning of “marriage” should be broadened. There are strong arguments on both sides. The point I am making is that each person who uses the word “marriage” should be prepared to state their definition.

The attitude towards Euclid’s axioms changed in the eighteenth century. They were no longer considered to be self-evident, but merely part of the definition of a particular geometry called Euclidean geometry. Other, slightly different, sets of axioms were set up for other geometries. From a mathematical point of view all of them are correct. It is up to the scientist, the physicist, the cosmologist, to decide which is correct for our universe. And the jury is still out on that question.

A rather different state of affairs exists for set theory. A “set” is a collection of “things”. In axiomatic set theory these things are mathematical objects. Now unlike group theory, where there are lots of systems satisfying the axioms, in axiomatic set theory we are attempting to describe a concept that we hold intuitively.

§2. The Russell Paradox

Set theory has come to underlie all of mathematics, so in a sense it is the foundation for all mathematics. Up to the end of the 19th century it was considered that the truths of set theory were self-evident, just as we don’t fuss too much about the logic we employ. One of the assumptions is that for any property that things might have there is a corresponding set, consisting of all the things that have that property. This is the process of turning an adjective into a noun. “Black” is an adjective, so there is the set of all black things. But the
philosopher Bertrand Russell, who was interested in the foundations of mathematics, pointed out that the set of all sets that do not belong to themselves is self-contradictory.

Perhaps a bit of notation will help us to understand this. The fundamental property of sets is that things belong to them. We denote the fact that the thing $x$ belongs to the set $S$ by the notation $x \in S$.

If $P$ is a property, like being black, and $x$ is a thing, we denote the statement that $x$ has the property $P$ by $Px$. So if $c =$ a crow and $Bx =$ “$x$ is black” then $Bc$ is a true statement, while $Bs$ is false if $x =$ a dove. Crows are black but doves are not.

The set that corresponds to the property $P$ is denoted by $\{x \mid Px\}$. Read it as “the set of all $x$ such that $Px$ (or $Px$ is true).” The naïve assumption was that for all properties $P$ there must be a set $\{x \mid Px\}$.

Russell considered the property of something not belonging to itself – in the sense of set belonging. Here the something is a set. A set can belong to another set because it is possible to have sets of sets, or sets of sets of sets .... If $T$ is the set of all pairs of distinct whole numbers then the set consisting of just 3 and 5 would belong to $T$. The symbol for “not belonging” is $\notin$, just like the symbol for “not equals” is obtained by crossing out the equals sign, as in $\neq$. Now some sets clearly don’t belong to themselves. The set of all positive numbers is not a positive number. But the set of all sets is a set.

So Russell said, what if $S = \{x \mid x \notin x\}$? This would be the set of all sets that are not members of themselves. This would be the case for most sets we might think of.

The set of all even numbers is not an even number. The set of all triangles is not a triangle. The set of all infinite sets is an infinite set.

The question is:

**Does $S$ belong to $S$?**

Clearly the answer would have to be either “yes” or “no”, but let’s consider each possibility in turn.

**SUPPOSE that $S \in S$.**
Then it must satisfy the corresponding property, that is $S \notin S$. This is a contradiction.

**SUPPOSE that $S \notin S$.**
Then $S$ satisfies the property that defines $S$ and so $S \in S$. Again, a contradiction.

This seems like one of those logical paradoxes like the sentence “THIS SENTENCE IS FALSE”. But we can’t ignore it. Under our naïve concept of set theory such a set exists. If we want to ban it from being a set we’d better explain to it why it’s being kicked out!

This may also remind you of the argument from the chapter on the uncountable. The difference is that in that case there was an assumption that led to the contradiction. If one can find a different chairman for every committee then we get a contradiction. Therefore it is
impossible to provide a different chairman for every committee. It is false that there is the same number of subsets as elements.

But with the Russell Paradox there appears to be no such initial assumption, apart from the intuitively obvious “fact” that for every property there’s a set of all things with that property. Well, then, intuitively obvious or not, this assumption has to go.

Here we have a fundamental contradiction in set theory. And since we want to build mathematics upon set theory, all of mathematics would fall to the ground if we didn’t remove such a flaw. If you allow a single contradiction into mathematics you can prove anything.

I remember one of my lecturers telling me this and when someone asked him to prove that he was the Pope, assuming that 0 = 1, he said, “If 0 = 1 then, adding 1 to both sides, then 1 = 2. The Pope and I are two people, so the Pope and I are the one person. QED.”

Well, you can imagine the fuss that Russell’s Paradox caused when it was first announced. At least it caused a fuss amongst those who were bothered about the foundations of mathematics. Ordinary working mathematicians just said, “oh, that’s interesting” and went back to their work. They knew that someone would fix up the problem, and that they did.

The way of fixing up the problem was to set up a collection of axioms that made some restrictions on which properties do lead to a set. There have been several formulations but they have all been proved to be equivalent to one another. The most well-known set of axioms are called the ZF axioms, after their proposers Zermelo and Fraenkel. I won’t list them here because they are long and sound quite technical. Basically they mostly say that “if such and such is a set the so and so is a set”. They are all dependent on already having some sets with which to make other sets - except for the first axiom, the existence of the empty set.

The empty set is the set with no elements. It doesn’t matter what the no elements are. The set of unicorns is the same empty set as the set of elves or the set of whole numbers lying strictly between 1 and 2. Axiom 1 in the ZF system says: There exists a set corresponding to the property \(x \neq x\), that is \(\{x \mid x \neq x\}\) exists. The symbol for the empty set is \(\emptyset\). Now you might be thinking that is silly to have a set with nothing in it.

“Oh, I have a collection of vintage Rolls Royce automobiles.”

“Wow! How many have you got?”

“Oh, it’s the empty set.”

Stupid as it might seem, where would we be without the number zero? For centuries zero was never considered a number. Why have a number for something that doesn’t exist. Yet, our modern system of notation for numbers relies on zero. The difference between my bank balance and that of Bill Gates is just a whole lot of zeros!

Now there’s something rather delightful in the fact that all of mathematics can be manufactured from the empty set. First there’s the set \(\{\emptyset\}\) that contains just the empty set. It isn’t the empty set itself because it does have something in it, even though that something is empty. Then there is \(\{\emptyset, \{\emptyset\}\}\). This set contains two sets, the empty set itself, and the set consisting of the empty set. It might seem that we’re splitting hairs here, but the distinction between \(\emptyset\) and \(\{\emptyset\}\) is important. In fact, when the number 2 is defined it is defined in this
way of developing mathematics, it is the set \{\varnothing, \{\varnothing\}\} and 3 is \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}. If this seems a rather esoteric way of defining the number 3, let me ask how you would define it. I’m sure what you might come up with would be more intelligible to a typical kindergarten pupil than \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} but it wouldn’t stand up to the high standard of rigour that professional mathematicians require.

You might say that this shows that God created mathematics. Just as God created the world from a void he created the whole of mathematics from the empty set! On the other hand, if you are somewhat of an atheist, at least you’ll find a resonance between mathematics created from the empty set and the big-bang theory of how the universe began.

§3. Axioms for Mathematics

Almost all of mathematics can be built up from the following axioms. They are called the Zermelo-Fraenkel Axioms, or ZF for short. Other foundations have been suggested, but they are all equivalent to the ZF creed. For “creed” it is—just as a religious creed. They are statements whose truths are taken without proof. One just has to believe in them. Remember that it is not possible to prove something from nothing.

In addition, there are assumptions about logic, we would be considering logical axioms as well. These will regulate the use of words such as “and”, “or” and “implies”.

Six of the eight ZF axioms are:

**Equality:** Two sets are equal if they have precisely the same elements.

**Empty Set:** There is a set with no elements.

**Pairs:** If \(S, T\) are sets there is a set with just \(S\) and \(T\) as elements.

**Powers:** If \(S\) is a set so is the set of all subsets of \(S\).

**Union:** If \(S\) is a set so is the set of all elements of elements of \(S\).

**Specification:** If \(S\) is a set and \(P\) is any property that can be expressed entirely in terms of set membership, then there is a set whose elements are precisely those elements of \(S\) for which the property holds.

The other two axioms are a bit more technical, so we’ll omit them. A full discussion can be found in my notes on Set Theory. On the basis of these eight axioms virtually the whole of mathematics can be built. (This is outlined in my Set Theory Notes.)

So can we now be assured that no further contradiction, like Russel’s Paradox will arise? This amounts to asking whether the ZF axioms are consistent. The slightly disturbing answer is that no, we do not know that they are consistent. Most mathematicians believe that they are, but most mathematicians believe that we will never be able to prove consistency.
§4. Consistency

A set of axioms is **inconsistent** if a contradiction can be validly derived from them. If it is not inconsistent then it is defined to be **consistent**. The easiest way to prove consistency is to come up with a model for the axioms, that is, an actual interpretation that satisfies all the axioms.

It’s easy to come up with an inconsistent set of axioms. For example consider the following axioms for a *super number*. The set of super numbers has two operations, called addition and multiplication, such that the following axioms hold.

**Axiom 1:** There is a super number 0, such that \( n + 0 = n \) for all super numbers, \( n \).

**Axiom 2:** There is a super number 1 such that \( 1 + 1 \neq 1 \).

**Axiom 3:** \( (x + y)z = xy + xz \) for all super numbers \( x, y \) and \( z \).

**Axiom 4:** There is a super number \( \infty \) such that \( 0 \infty = 1 \).

This system of axioms is inconsistent. Here’s a proof.

By axiom 1, \( 0 + 0 = 0 \), and so \( (0 + 0)\infty = 0\infty \).

By axiom 3, \( 0\infty + 0\infty = 0\infty \).

By axiom 4, \( 1 + 1 = 1 \), contradicting axiom 2.

Here’s another axiomatic system. There is a set of undefined things called “persons” and three undefined relations:

- \( X \) is the father of \( Y \),
- \( X \) is the mother of \( Y \),
- \( X \) is married to \( Y \).

Now the terminology suggests we’re really thinking of family relationships, and certainly that is what we may have had in mind. But it must be emphasized that these things are undefined and so we must not make any use of what we know of actual family relationships.

We assume the following axioms:

**Axiom 1:** Each person has a unique mother and a unique father.

**Axiom 2:** No father can be a mother.

**Axiom 3:** No person can be its own father or mother.

**Axiom 4:** If two people have the same mother then they have the same father.

**Axiom 5:** Everyone’s mother and father are married.

**Axiom 6:** Persons with the same mother can’t marry.

You may disagree with some of these axioms. Certainly axioms 4 and 5 don’t hold with today’s complicated family relationships. But all that means is that this axiomatic system doesn’t fit family relationships in our society. But it might fit some ideal community. Remember that all that can be asked of an axiomatic system is that it be consistent. If it doesn’t fit some particular interpretation then that’s a separate matter altogether.
We define a **model** of a set of axioms to be a specific example that fits the axioms. Although we were motivated by family relationships in the choice of our terminology we have seen that this example is not a model. Not all the axioms are true for families.

On the other hand the following is indeed a model. In this model the persons are the numbers 1, 2, 3, 4, 5 and 6. The mothers fathers and spouses are given by the following table:

<table>
<thead>
<tr>
<th>MODEL A</th>
<th>mother</th>
<th>father</th>
<th>spouse</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>–</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

It may seem weird to think of family relationships like these between numbers, but that is only because of the language we have chosen. If we are to work with this model then we have to empty our minds of anything we know, from our experience of families, about family relationships. We can only build our theory based on the axioms.

So is this example really a model? The axioms can be readily checked.

The fact that there is a model that satisfies the axioms ensures that the set of axioms is **consistent**. It is never possible to obtain a contradiction from a consistent set of axioms.

Once we set up a set of axioms we make certain definitions and then prove theorems by making logical inferences from these axioms. Once we establish a few theorems we prove further theorems based on previously proved theorems. We consider these theorems to be true in any model, though you must remember that they are only true if the axioms are.

Suppose, in our example, we define a **parent** to be a “person” who is either a mother or a father and a **grandmother** to be the mother of a parent.

**Theorem 1:** Every person has exactly two grandmothers.

**Proof:** A person \( P \) has his mother’s mother and his father’s mother and so by axiom 1 has either one or two grandmothers. But what if they were the same? Then \( P \)’s mother and father would have the same mother and so by axiom 4 they would have the same father. By axiom 5 they must be married. But by axiom 6 they can’t be married, a contradiction.

This theorem will be true for an ideal community where only married people have children, and where no-one ever remarries. But it is also true in our numerical model. In fact it is true for every model that fits these axioms.

An inconsistent set of axioms is totally useless because there can be no example. So, super numbers don’t exist. Only consistent axioms need apply!

In axiomatic set theory we often consider extra “optional axioms”. There is a song called *I’m My Own Grandpa!*
I'M MY OWN GRANDPA!
Many, many years ago when I was twenty-three
I was married to a widow who was as pretty as could be
This widow had a grown-up daughter who had hair of red
My father fell in love with her and soon they too were wed.

This made my dad my son-in-law and really changed my life.
For now my daughter was my mother, ‘cause she was my father’s wife.
And to complicate the matter, even though it brought me joy
I soon became the father of a bouncing baby boy.

My little baby boy then became a brother-in-law to Dad
And so became my uncle, though it made me very sad,
For if he were my uncle, then that also made him brother
Of the widow’s grown-up daughter, who was of course my step-mother.

Father’s wife then had a son who kept them on the run,
And he became my grandchild, for he was my daughter’s son.
My wife is now my mother’s mother and it makes me blue
Because although she is my wife, she’s my grandmother too.

Now if my wife is my grandmother, then I’m her grandchild,
And every time I think of it, it nearly drives me wild.
‘Cause now I have become the strangest case you ever saw
As husband of my grandmother, I’m my own grandpa.

I’m my own grandpa
It sounds funny, I know but it is really so
I’m my own grandpa.

[This was written by Latham Dwight and Jeff Moe and was published by Colgems-EMI Music.]

Such a complicated family should be not allowed in the ideal family model so let’s consider the following additional axiom.

Axiom 7: No person is their own grandfather.

This seems reasonable enough, and would be true in our world of ideal families where nobody ever remarried. However it is false for our numerical model since 3 is his own grandfather by virtue of the fact that 3’s mother is 6 and 6’s father is 3.
What this example means is that Axiom 7 cannot be proved from the other six axioms, because the model satisfies those six axioms but not the last. We say that Axiom 7 is **independent from** the first six axioms.

On the other hand, our idealised concept of idealised families does satisfy all seven axioms. So we say that Axiom 7 is **consistent with** the first six axioms.

The definition of “idealised families” is a little vague, and in any case there is doubt as to who was Adam’s mother, so consider a different numerical model.

<table>
<thead>
<tr>
<th>MODEL B</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td>5,7</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>3,9</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>7</td>
<td>2</td>
<td>5,9</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>5,7</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>9</td>
<td>4</td>
<td>3,11</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>5</td>
<td>7</td>
<td>3,9</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>9</td>
<td>6</td>
<td>3,11</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>3</td>
<td>9</td>
<td>5,7</td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td>11</td>
<td>8</td>
<td>5,7</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>5</td>
<td>11</td>
<td>3,9</td>
</tr>
<tr>
<td>11</td>
<td>6</td>
<td>7</td>
<td>10</td>
<td>5,9</td>
</tr>
</tbody>
</table>

This clearly satisfies all seven axioms. So Axiom 7 is independent of, but consistent with, the other six axioms. We say that it is **undecidable**. That is it is impossible to prove it from the other six axioms (because of model A). On the other hand it is impossible to prove that it is false (because of model B).

As mentioned earlier, a fundamental question that should be asked is whether these ZF axioms are consistent. The rather unsettling answer is, we don’t know. To prove that the ZF axioms are consistent we’d have to come up with a model. The problem is that the mathematical objects we would need to use the ZF axioms to construct such a model.

The general belief among mathematicians is that they will never be proved to be consistent but we hope, indeed we believe, they are. If someone ever comes up with a contradiction then the relevant people will fix the problem and the whole of mathematics will continue without a hiccup. In one sense mathematics rests on the foundations of set theory – in another sense it doesn’t. Mathematicians have a fervent belief in the consistency and validity of mathematics as unshakable as a religious belief. “I believe, because I know!”

§5. The Axiom of Choice

Now, what is really interesting is that there a few things that can’t be proved from the ZF axioms which most mathematicians believe are true. One of these is the Axiom of Choice, abbreviated to AXC. In a nutshell the AXC says that if you have a whole bunch of non-empty sets you can simultaneously choose one thing out of each of them. This seems an obvious enough statement, but remember that it says that this is possible, even if the sets are infinite and even if there are infinitely many of them.
Of course such a choice is impossible in practice because it would take infinite time, but we’re not talking about “in practice”. The question is, does such a choice exist and can they choices form a set? (The last question is not quite the one that is asked, but it’s near enough for our purposes.)

The Axiom of Choice has been proved to be consistent with, and independent of, the ZF axioms. To show this you assume the ZF axioms and construct a model in which not only the ZF axioms hold, but also the Axiom of Choice. That is the “consistent with” part. Then you construct a different model, with a different definition of “belonging to” that satisfies the ZF axioms but does not satisfy the Axiom of Choice. That is the “independent of” half of the statement. Putting these halves together we come up with the statement:

**THE AXIOM OF CHOICE IS UNDECIDABLE.**

This means that, assuming the ZF axioms are consistent, you will never be able to prove that the AXC is true. But nor will you ever be able to prove that it is false. If ever a contradiction arises in mathematics when using the Axiom of Choice it won’t be the fault of that axiom. It will mean that an inconsistency will have been found in the ZF axioms themselves. If ever a contradiction arises from denying the Axiom of Choice it will mean that the ZF axioms themselves are inconsistent, not the denial itself.

The bottom line is that you are free to choose! You can believe in AXC or not. Both positions are logically valid. Naturally, like most mathematicians, you will no doubt opt to believe in AXC. It sounds so plausible. But before you become a paid-up member of the Axiom of Choice religion, let me point out the following consequence of the Axiom of Choice.

It has been proved, assuming the ZF axioms, together with the AXC, that in principle it is possible to take a solid ball and dissect it into several pieces and to reassemble the pieces to make two solid balls of the same size as the original one!

Your reaction to this is probably to say that this proves that the AXC is false. After all, such a situation would contradict the law of conservation of volume, surely. If you take a piece of wood its volume would remain constant no matter how you cut it up and reassembled the pieces. That is, ignoring the sawdust which, of course, we are doing.

However the law of conservation only applies if the pieces have a defined volume. If a set of points is highly fragmented, like a cloud of infinitely small particles, then it is not possible to define its volume.
The way of dissecting the original sphere and reassembling them is not something one could replicate, even with precision tools. If it was possible to convert one ounce of gold into two with a laser cutter, the price of gold would plummet! The “pieces” that are required to perform this magic are so highly fragmented that their volumes don’t exist.

Needless to say, while most mathematicians are happy to accept the Axiom of Choice, because it simplifies the statements of many of their theorems, there is a determined minority who reject it. A comforting thought is that no bridge will ever fall down because its engineer believed or didn’t believe in the AXC.

The difference between believing or not believing is more aesthetic than practical. In this sense it is rather different to a religious belief. The Axiom of Choice believers will never wage war on the infidels, and no mathematician will become a martyr to his or her belief. The general consensus is that one should try not to use the Axiom of Choice, but if necessary one uses it, and admits that it is “on the basis of the Axiom of Choice”.

§6. The Peano Axioms

The very first mathematical system we encountered was the system of the natural numbers 0, 1, 2, 3, … When we did so, in kindergarten or even before, we were not interested in precise definitions. We learnt the many properties of natural numbers on the authority of our parents and teachers. Nowhere did we see a definition of the number 2, or a precise proof of the fact that 1 + 1 = 2. We might have experimented with a few pairs of objects and observed that combining one with another we got a collection which, when we counted, gave us 2. Hence we learnt our mathematics as an experimental science.

Of course we did notice that sometimes it didn’t work. Pour a litre of water into a bowl containing a litre of sugar and you will find you get a whole lot less than a litre of sugar syrup. This can be explained, in part, by the air spaces between the grains of sugar, but to account for the reduction in volume completely you need to take the chemistry of solutions into account. Nevertheless you understood that something different is going on here and that 1 + 1 = 2 is still valid mathematically.

One approach to constructing the natural numbers, and their arithmetic, rigorously is to build them up as sets of sets of sets within axiomatic set theory. Another approach is to define them by a set of axioms, the Peano Axioms.

We postulate a set of undefined things, together with an undefined function “successor”. You can think of the “successor” of $n$ as $n + 1$, written $n'$, but that interpretation isn’t specifically part of the axioms.
Axiom 1: 0 is a natural number;
Axiom 2: If \( n \) is a natural number then so is its successor \( n^+ \);
Axiom 3: There is no natural number \( n \) for which \( n^+ = 0 \);
Axiom 4: If \( S \) is any set of natural numbers that contains 0, and contains the successor of \( n \) whenever it contains \( n \), then \( S \) is the set of all natural numbers.

On the basis of these axioms we can define addition and multiplication and prove the basic properties of arithmetic.

§7. Gödel’s Theorem

Gödel proved that if you have any set of axioms for the natural numbers there are true statements that cannot be proved from the axioms. We say that arithmetic is incomplete. Indeed he proved that the same is true for any axiomatic system in which the natural numbers can be formulated. It is a very long and technical proof – one that we won’t go into details here.

At the heart of the proof is a very clever method for converting statements about the system into arithmetic statements within the system. For a start, statements are expressed symbolically, such as \( \forall x(\neg (x = 0) \rightarrow \exists y(xy = 1)) \) which means “for all \( x \), if \( x \) is not equal to zero then there exists \( y \) such that \( x \) times \( y \) is equal to 1”. Gödel devised a system for coding these statements as a number by assigning a code to each symbol and building up a number for each statement on the basis of that. So, given a number \( n \) one could, if that \( n \) indeed represented a statement, decode it and so obtain the corresponding statement \( G(n) \). Every statement would have a code, but not every code would correspond to a valid statement. The numbers involved would be extremely large, but as this is an “in principle” exercise, that isn’t a worry.

Now consider the statement that a given statement \( S \) is provable. A proof is just a list of statements, where each one is an axiom, or a previously proved theorem, or a logical consequence of the previous ones, and where the statement of the theorem is the last in the list. There is a mechanical way of testing the validity of a proof and so one could, in principle, write a computer program for testing whether a given statement is provable from the axioms. It would be a case of generating all possible lists of statements that have \( S \) as the last statement, and then testing the “proof” for validity.

Gödel showed how provability could be expressed as an arithmetic statement about natural numbers and so the statement \( P(n) = \text{“the statement with Gödel number } n \text{ is provable”} \) can be expressed as an arithmetic statement and so will have a certain Gödel number. Similarly, the statement \( N(n) = \text{“the statement with Gödel number } n \text{ is not provable”} \) has a Gödel number, say \( g \).

Gödel then asked whether \( N(g) \) is true or false. The statement \( N(g) \) claims that it, itself, is unprovable. Thus we can obtain, as a purely arithmetic statement within the language of arithmetic, a statement which claims “I am unprovable”. Now such a statement cannot be false because being false would mean it was provable and hence true. It must therefore be true and hence it’s a true but unprovable statement in arithmetic. But wait! Haven’t we just proved that it is true?
Certainly we gave a meta-mathematical proof. But this proof is not one which could be expressed as an arithmetic proof within the system. Our unprovable statement is not unprovable in any absolute sense. It might not even be meaningful to talk about absolute unprovability. N(g) is unprovable in the relative sense that no proof of it could ever be constructed which starts from the axioms and proceeds using the rules of inference. And even if the axioms and rules were supplemented by others, so long as they remained finite in number, the existence of unprovable statements would remain.

**JOKE: PALINDROME**

An Englishman, an Irishman and a Scotsman go into a bar. An American, who was already in the bar comes up to the Englishman and says, “Hey buddie, if you can tell me a good joke I’ll buy you a beer”.

So the Englishman clears his throat and says, “398627689560145”. At this the bar erupts into an uproar of laughter. The American looks puzzled, but says, “well it appears that was a great joke, so what’ll you have?”

A little later the American goes up to the Scotsman and says, “I’ll buy you a whisky if you can beat that last joke”. The Scotsman stands on a stool, adjust his kilt and says, “490284930613043”. Once again the bar erupts into laughter, even louder than before. Several patrons are so carried away by their laughter that they roll around on the floor. So the American buys the Scotsman a Scotch.

A little later the American turns to the Irishman and says, “You Irish are renowned for your wonderful humour. I bet you can top that last joke – if you do, I’ll buy you a pint of Guinness”.

So the Irishman jumps up on the bar, adjusts his cap, and says, “123456787654321”. There is deathly silence. Not even a murmur is heard. The American looks puzzled.

“I’ve worked out that you folks must number your jokes so that all you have to do is to give the joke number and you all know what the joke is. But back in the States we tell our jokes in full. Now I’m a little puzzled why that last joke fell so flat. What went wrong?”

“Ah”, says the Scotsman, “you know what the Irish are like. They’re always getting things back to front.”

“Well”, said the American, “would you mind telling me that last joke in full”.

“Och, aye”, said the Scotsman, “but are ye sure ye want to hear it. As I said it’s not ry funny”.

“Well, yes”, said the American, “I’m fascinated by British humour”.

“OK”, says the Scotsman, “Joke number 123456787654321 goes like this. An Englishman, an Irishman and a Scotsman go into a bar. An American, who was already in the bar comes up to the Englishman and says, “Hey buddie, if you can tell me a good joke I’ll buy you a beer”. So the Englishman clears his throat and says, “398627689560145”.

132