DERIVATORS: DOING HOMOTOPY THEORY WITH 2-CATEGORY THEORY

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Overview:

1. *From 2-category theory to derivators.* The goal here is to motivate the definition of derivators starting from 2-category theory and homotopy theory. Some homotopy theory will have to be swept under the rug in terms of constructing examples; the goal is for the definition to seem natural, or at least not unnatural.
2. *The calculus of homotopy Kan extensions.* The basic tools we use to work with limits and colimits in derivators. I'm hoping to get through this by the end of the morning, but we'll see.
3. *Applications:* why homotopy limits can be better than ordinary ones. Stable derivators and descent.

References:

- http://ncatlab.org/nlab/show/derivator — has lots of links, including to the original work of Grothendieck, Heller, and Franke.
- http://arxiv.org/abs/1112.3840 (Moritz Groth) and http://arxiv.org/abs/1306.2072 (Moritz Groth, Kate Ponto, and Mike Shulman) — these more or less match the approach I will take.

1. Homotopy theory and homotopy categories

One of the characteristics of *homotopy theory* is that we are interested in categories where we consider objects to be "the same" even if they are not isomorphic. Usually this notion of sameness is generated by some non-isomorphisms that exhibit their domain and codomain as "the same". For example:

(i) Topological spaces and homotopy equivalences
(ii) Topological spaces and *weak* homotopy equivalence
(iii) Chain complexes and chain homotopy equivalences
(iv) Chain complexes and quasi-isomorphisms
(v) Categories and equivalence functors

Generally, we call morphisms like this **weak equivalences**.

1.1. **Homotopy limits.** The problem is that standard categorical constructions, like limits and colimits, do not respect this weaker notion of sameness. This is not usually a problem with products and coproducts: you can check for instance that a product or coproduct of homotopy equivalences is again such. But it becomes a problem for pullbacks and pushouts.

- A disc D^2 is homotopy equivalent to a point $*$. But the pushout of the left diagram is a 2-sphere, while the pushout of the right diagram is a point, and these are not homotopy equivalent.

$$\begin{array}{ccc} S^1 & \longrightarrow & D^2 \\ \downarrow & & \\ D^2 & & \end{array} \qquad\qquad \begin{array}{ccc} S^1 & \longrightarrow & * \\ \downarrow & & \\ * & & \end{array}$$

We can solve this by constructing things called *homotopy limits and colimits*.

Definition 1.1. The **homotopy pushout** of a span of spaces

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \\ B & & \end{array}$$

is the space

$$\Big(B \sqcup C \sqcup (A \times [0,1])\Big) \Big/ \big(f(a) \sim (a,0) \text{ and } g(a) \sim (a,1)\big)$$

Definition 1.2. The **homotopy pullback** of a cospan of spaces

$$\begin{array}{ccc} & & C \\ & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[f]{} & D \end{array} \tag{1.3}$$

is the space

$$\Big\{(b,c,\delta) \;\Big|\; b \in B, c \in C, \delta : [0,1] \to D, \delta(0) = f(b), \delta(1) = g(c)\Big\}$$

This works, but it sets us back to the world before category theory! We have to manipulate explicit constructions, rather than characterizing things by universal properties.

1.2. **Homotopy categories.** One thing you might try naively is to force the weak equivalences to *become* isomorphisms.

Definition 1.4. If $\mathscr{C}$ is a category with a collection $\mathscr{W}$ of "weak equivalences", its **homotopy category** $\mathscr{C}[\mathscr{W}^{-1}]$ is the universal category with a map from $\mathscr{C}$ in which the weak equivalences become isomorphisms. In other words, we have a functor $\gamma : \mathscr{C} \to \mathscr{C}[\mathscr{W}^{-1}]$ such that for any category $\mathscr{D}$, the functor

$$\mathcal{C}at(\mathscr{C}[\mathscr{W}^{-1}], \mathscr{D}) \xrightarrow{-\circ\gamma} \mathcal{C}at(\mathscr{C}, \mathscr{D})$$

is an isomorphism onto the full subcategory of functors that send $\mathscr{W}$ to isomorphisms in $\mathscr{D}$.

Explicitly, the morphisms in $\mathscr{C}[\mathscr{W}^{-1}]$ can be described as zigzags

$$X \leftarrow X_1 \to X_2 \leftarrow X_3 \to \ldots \leftarrow X_n \to Y$$

where the backwards-pointing arrows are in $\mathscr{W}$ (representing the formal inverses of those arrows). We then have to quotient these zigzags by some equivalence relation.

One problem with this is that the hom-sets of $\mathscr{C}[\mathscr{W}^{-1}]$ may no longer be small even if those of $\mathscr{C}$ are. There are ways to deal with this, generally along the following lines:

Proposition 1.5. *If $\mathscr{C} = \mathbf{Top}$ and $\mathscr{W}$ is the homotopy equivalences, then $\mathscr{C}[\mathscr{W}^{-1}]$ is isomorphic to the category with objects from $\mathscr{C}$ and*

$$\mathscr{C}[\mathscr{W}^{-1}](X,Y) = \mathbf{Top}(X,Y)/\sim$$

Proof. Claim $F : \mathscr{C} \to \mathscr{D}$ inverts $\mathscr{W}$ iff it identifies homotopic maps. "If" is obvious; for "only if", a homotopy $f \sim g : X \to Y$ is a diagram

$$\begin{tikzcd} X \arrow[dr, "i_0"'] \arrow[drr, bend left, "f"] & & \\ & X \times [0,1] \arrow[r, "H"] & Y. \\ X \arrow[ur, "i_1"] \arrow[urr, bend right, "g"'] & & \end{tikzcd}$$

Both maps $i_0, i_1 : X \to X \times [0,1]$ are split monos with a common retraction r (the projection) and homotopy equivalences. Thus, if F inverts them, it also identifies them, since $F(i_0)$ and $F(i_1)$ are both inverse to $F(r)$. Henc $F(f) = F(Hi_0) = F(Hi_1) = F(g)$. □

In fancier examples, we have to first restrict to a subcategory (e.g. CW-complexes, chain complexes of projectives or injectives) before quotienting by homotopy.

Unfortunately, the homotopy category does not solve the problem of homotopy limits: homotopy limits are *not* (in general) limits in the homotopy category! It is usually true of products and coproducts, which generally require very little "homotopification". But consider the homotopy pullback P of (1.3), which comes with a homotopy commutative square

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\sim"] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & D. \end{tikzcd}$$

This is, in particular, a commutative square in the homotopy category, and indeed any other homotopy commutative square

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "h"'] \arrow[dr, phantom, "H"] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & D \end{tikzcd}$$

yields a map $X \to P$ defined by $x \mapsto (h(x), k(x), H(x))$ which makes the appropriate diagrams commute. However, this map is not in general *unique*, because we had to *choose* a particular homotopy H in order to define it, while a commutative square in the homotopy category does not come *equipped* with such an H.

Exercise: find an explicit counterexample. It's probably easiest to work with $\mathcal{C}at$ (in fact, groupoids suffice) rather than spaces.

1.3. **Abstract homotopy theory.** There are a number of abstract frameworks for homotopy theory:

(1) *Quillen model categories* and related ideas. These are a collection of tools that make it easier to work with things like homotopy limits concretely, as above. But it still doesn't make them "categorical".
(2) $(\infty,1)$*-categories.* In $\mathcal{C}at$ with equivalences of categories, the obvious solution is to consider it as a 2-category rather than a 1-category, with 2-dimensional limits. Similarly, we can make spaces into an ∞-category, with homotopies and higher homotopies and ∞-limits. Here homotopy limits have true universal properties, but the notion of "∞-category" is quite technically complicated, as is the definition of the appropriate universal property.
(3) *Triangulated categories* and their ilk. Here we consider the homotopy category equipped with the structure of certain "weak" limit-notions that have existence but not uniqueness.
(4) *Homotopy type theory.* This is an "internal language" for certain model categories and $(\infty,1)$-categories. Come to my talk at CT.
(5) *Derivators.* Here we equip the homotopy category with more data, which enables us to characterize homotopy limits by *actual*, *ordinary* (not higher-categorical), universal properties.

Each has advantages and disadvantages, and tells part of the story of homotopy theory. Some things can be done equally well with any of them, others are easier in one or the other. Today I'll talk about derivators, which have a lot of advantages: they don't require very much machinery, and they are quite powerful and flexible. I will point out as we go how they connect to the other frameworks.

2. Prederivators

Suppose we were raised steeped in 2-category theory, as Richard described a couple days ago, and someone told us for the first time about homotopy theory. What would we think of?

One thing we would hopefully notice quickly is that the homotopy category is a 2-colimit: a **coinverter**. Given $\mathscr{C}$ with weak equivalences $\mathscr{W}$, let $\overline{\mathscr{W}}$ denote the full subcategory of $\mathscr{C}^2$ whose objects are $\mathscr{W}$. Then we have a 2-cell

$$\overline{\mathscr{W}} \underset{\text{cod}}{\overset{\text{dom}}{\underset{\Downarrow}{\rightrightarrows}}} \mathscr{C}$$

of which $\mathscr{C}[\mathscr{W}^{-1}]$ is the coinverter.

Now suppose our friend tells us that homotopy categories are bad. That is, they lose too much information: they don't let us characterize the limit-constructions we want. We know that in general, a way to get "good" colimits is to *freely adjoin* them, that is, to apply the Yoneda embedding and take colimits in the presheaf category. Thus, we might consider doing this for coinverters.

Just to be a little bit careful about size, let $\mathcal{C}at$ be the 2-category of small categories, and $\mathcal{CAT}$ the 2-category of large ones (which is itself even larger). And while I'm mentioning notation, let $\mathbb{2} = (0 \to 1)$ be the interval category, and $\mathbb{1}$ the terminal category (there are too many 1's in this subject). For the same reason, I'll write id for identity maps.

Our $\mathscr{C}$ will be large, but we consider only the restricted Yoneda embedding

$$y : \mathcal{CAT} \hookrightarrow [\mathcal{C}at^{\mathrm{op}}, \mathcal{CAT}].$$

That is, given a category $\mathscr{C}$ we associate to it the 2-functor $y(\mathscr{C}) : \mathcal{C}at^{\mathrm{op}} \to \mathcal{CAT}$ defined by $y(\mathscr{C})(A) \coloneqq \mathscr{C}^A$.

Definition 2.1. A **prederivator** is a 2-functor $\mathcal{C}at^{\mathrm{op}} \to \mathcal{CAT}$. The 2-category $\mathcal{PDER}$ of prederivators has pseudonatural transformations as morphisms.

The prederivator $y(\mathscr{C})$ is called the **represented prederivator** at $\mathscr{C}$. We likewise have $y(\overline{\mathscr{W}})$, and we may consider the coinverter of

$$y(\overline{\mathscr{W}}) \underset{\mathrm{cod}}{\overset{\mathrm{dom}}{\Downarrow}} y(\mathscr{C})$$

As always in a presheaf category, colimits are pointwise. Thus, if we let $\mathscr{H}o(\mathscr{C})$ denote this coinverter, we have

$$\mathscr{H}o(\mathscr{C})(A) = \mathscr{C}^A \left[(\mathscr{W}^A)^{-1}\right].$$

In other words, $\mathscr{H}o(\mathscr{C})(A)$ is the homotopy category of the diagram category $\mathscr{C}^A$ with the *pointwise weak equivalences* inverted. We call it the **homotopy prederivator** of $\mathscr{C}$. Note that $\mathscr{H}o(\mathscr{C})(\mathbb{1}) = \mathscr{C}[\mathscr{W}^{-1}]$ is the ordinary homotopy category of $\mathscr{C}$. Functoriality gives us a functor $u^* : \mathscr{H}o(\mathscr{C})(B) \to \mathscr{H}o(\mathscr{C})(A)$ for any $u : A \to B$.

It is very important to note that $\mathscr{H}o(\mathscr{C})(A)$ is different from $\big(\mathscr{C}[\mathscr{W}^{-1}]\big)^A$. We have a functor from one to the other:

$$A \to \mathcal{C}at(\mathbb{1}, A) \xrightarrow{\mathscr{H}o(\mathscr{C})} \mathcal{CAT}\big(\mathscr{H}o(\mathscr{C})(A), \mathscr{H}o(\mathscr{C})(\mathbb{1})\big)$$

yields by exponential transpose

$$\mathscr{H}o(\mathscr{C})(A) \to \mathscr{H}o(\mathscr{C})(\mathbb{1})^A.$$

Indeed, this used only the 2-functoriality of $\mathscr{H}o(\mathscr{C})$, so it is true for any prederivator $\mathscr{D}$:

$$\mathscr{D}(A) \to \mathscr{D}(\mathbb{1})^A.$$

We call this the **underlying diagram** functor. It is *not* an equivalence, but it has some equivalence-like properties.

Consider the homotopy prederivator of spaces, with the simplest nontrivial case of $A = \mathbb{2}$.

- The objects of $\mathscr{H}o(\mathbf{Top})(\mathbb{2})$ are triples (X, Y, f) of two spaces and a continuous map between them.
- The objects of $\mathscr{H}o(\mathbf{Top})(\mathbb{1})^{\mathbb{2}}$ are triples $(X, Y, [f])$ of two spaces and a *homotopy class* of continuous maps between them.

Thus the underlying diagram functor is *essentially surjective*, since every homotopy class contains some map.

- The morphisms of $\mathscr{H}o(\mathbf{Top})(\mathbb{2})$ from (X, Y, f) to (X', Y', f') are obtained from (strictly) commutative squares by inverting squares that are levelwise homotopy equivalences.
- The morphisms of $\mathscr{H}o(\mathbf{Top})(\mathbb{1})^{\mathbb{2}}$ are homotopy-commutative squares (without specified homotopy).

Of course, not every homotopy-commutative square is strictly commutative, but the underlying diagram functor here is still *full*. To see this, suppose

$$(*)\qquad \begin{array}{ccc} X & \xrightarrow{[g]} & X' \\ {\scriptstyle [f]}\downarrow & & \downarrow{\scriptstyle [f']} \\ Y & \xrightarrow[{[k]}]{} & Y' \end{array}$$

is a homotopy-commutative square, and *choose* a homotopy H as well as representatives for g and h. Define Z to be the space

$$\Big(Y \sqcup (X \times [0,1])\Big) \Big/ ((x,1) \sim f(x)).$$

Define $p : Z \to Y$ by $p(y) = y$ and $p(x) = f(x)$. Then p is a homotopy equivalence, and we have a zigzag

$$\begin{array}{ccccc} X & \xleftarrow{\mathrm{id}} & X & \xrightarrow{g} & X' \\ {\scriptstyle f}\downarrow & & {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \longleftarrow & Z & \xrightarrow[H]{} & Y' \end{array}$$

whose image in $\mathscr{H}o(\mathbf{Top})(\mathbb{1})$ is $(*)$.

However, the underlying diagram functor is not *faithful*, because we had to choose a particular homotopy H to construct this lifting, and in general there might be many such. But it is *conservative*: any morphism in $\mathscr{H}o(\mathbf{Top})(\mathbb{2})$ that becomes an isomorphism in $\mathscr{H}o(\mathbf{Top})(\mathbb{1})^2$ was already an isomorphism. This is almost obvious: the isomorphisms in $\mathscr{H}o(\mathbf{Top})(\mathbb{1})^2$ are homotopy-commutative squares whose horizontal maps are isomorphisms in $\mathscr{H}o(\mathbf{Top})(\mathbb{1})$, hence homotopy equivalences, and we chose the weak equivalences pointwise in $\mathbf{Top}^{\mathbb{2}}$. However, the morphisms in $\mathscr{H}o(\mathbf{Top})(\mathbb{2})$ are actually *zigzags* of morphisms in $\mathbf{Top}^{\mathbb{2}}$, so something extra is required (it suffices to know that zigzags of length three, $\xleftarrow{\sim}\to\xleftarrow{\sim}$ suffice).

Definition 2.2 (Riehl–Verity)**.** A **(weakly) smothering functor** is a functor that is (essentially) surjective on objects, full, and conservative.

Weakly smothering functors have the important property that they reflect the relation of isomorphism between objects.

The category $\mathbb{2}$ is, in fact, a little misleading here: not every category A has the property that $\mathscr{H}o(\mathbf{Top})(A) \to \mathscr{H}o(\mathbf{Top})(\mathbb{1})^A$ is essentially surjective and full (though it is always conservative). What makes $\mathbb{2}$ special is that it's freely generated by a graph: it's not always true that a homotopy commutative diagram can be rectified. Consider, for instance, $A = B\mathbb{Z}_2$, the category with one object that has one nonidentity involution. Then an object of $\mathscr{H}o(\mathscr{C})(\mathbb{1})^A$ is a space with a map $f : X \to X$ such that $f \circ f$ is homotopic to the identity. Choosing any such homotopy yields two different homotopies from $f \circ f \circ f$ to f. If it came from an object of $\mathscr{H}o(\mathscr{C})(A)$ then the homotopy would be the identity, so in particular these two homotopies would be equal, hence trivially homotopic — and the latter property is preserved by homotopy equivalence. But there is no way in $\mathscr{H}o(\mathscr{C})(\mathbb{1})^A$ to be sure of that.

This discussion suggests a useful intuition: we can think of $\mathscr{H}o(\mathscr{C})(A)$ as the homotopy category of *homotopy coherent A-shaped diagrams* in $\mathscr{H}o(\mathscr{C})$, with homotopy coherent natural transformations between them. In a coherent diagram, not only does every diagram commute up to a specified homotopy, these homotopies have to be compatible up to higher homotopies, and so on. The non-surjectivity of the underlying diagram functor from some A then means that we cannot always choose homotopies coherently.

In fact, there's a theorem that (in good cases), any homotopy *coherent* diagram can be rectified to an equivalent strictly commutative one, so this intuition is valid. But I'm not going to explain that theorem, because it takes us more into the realm of model categories and $(\infty, 1)$-categories.

Take-away:

(i) A prederivator $\mathscr{D}$ comes with an *underlying category* $\mathscr{D}(\mathbb{1})$, and also a category $\mathscr{D}(A)$ that we should think of as consisting of *coherent A-shaped diagrams*, for all $A \in \mathcal{C}at$.
(ii) We have an underlying diagram functor $\mathscr{D}(A) \to \mathscr{D}(\mathbb{1})^A$, enabling us to draw an object of $\mathscr{D}(A)$ as if it were an ordinary diagram, with objects $X_a \in \mathscr{D}(\mathbb{1})$ for $a \in A$ and morphisms $X_f : X_a \to X_b$ for $f : a \to b$ in A.
(iii) In general, a coherent diagram is not determined, even up to isomorphism, by its underlying diagram.

3. Semiderivators

We've changed perspective from the category $\mathscr{C}[\mathscr{W}^{-1}]$ to the "presheaf" $\mathscr{H}o(\mathscr{C})$. What is the first thing you should ask about a presheaf?

Well, one of the first things is "is it a sheaf?" Or more generally, "what colimits does it preserve?" (or rather, what colimits does it take to limits, since it is contravariant). Of course, since these are 2-categories, we mean 2-colimits — but it would be most reasonable to ask only about preserving them up to equivalence rather than isomorphism.

We've already seen one colimit that $\mathscr{H}o(\mathscr{C})$ *almost* preserves: the category A is the copower (tensor) of $\mathbb{1}$ by A, and the comparison map asking whether $\mathscr{H}o(\mathscr{C})$ preserves this copower is precisely the underlying diagram functor

$$\mathscr{H}o(\mathscr{C})(A) \to \mathscr{H}o(\mathscr{C})(\mathbb{1})^A.$$

We've seen that this functor is not an equivalence, but when $A = \mathbb{2}$, it is weakly smothering. More generally, the copower of B by $\mathbb{2}$ is also almost preserved, i.e. the functor

$$\mathscr{H}o(\mathscr{C})(B \times \mathbb{2}) \to \mathscr{H}o(\mathscr{C})(B)^{\mathbb{2}}$$

is also weakly smothering.

A kind of colimit which not-too-surprisingly *is* preserved up to equivalence — even up to isomorphism, if we are careful enough — is coproducts. A diagram on a coproduct $\coprod_i A_i$ is just a family of diagrams on each A_i, and inverting some morphisms doesn't really change that. (That's not a proof, but I'll leave it as an exercise.)

One more colimit that is preserved is one that you may not think of as a colimit. First recall that *being monic* is a limit condition: a morphism $X \to Y$ in a category is monic iff

$$\begin{array}{ccc} A & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

is a pullback, i.e. any pair of maps $X \rightrightarrows A$ which become equal in B must factor uniquely through A, hence be equal. Thus, if a presheaf preserves colimits, then it takes epis to monos.

Analogously, in a 2-category, the property of *being conservative* is a (bi)limit condition: $X \to Y$ is conservative iff X is the limit of $X \to Y$ weighted by $\mathbb{2} \to \mathbb{I}$. (Exercise.) Thus, we can ask whether $\mathscr{H}o(\mathscr{C})$ takes conservatives in $\mathcal{C}at^{\mathrm{op}}$ — which are called *liberals* in $\mathcal{C}at$— to conservatives in $\mathcal{CAT}$.

Another exercise: a functor $u : A \to B$ is liberal iff every object of B is a retract of some object of A. In this case, $\mathscr{H}o(\mathbf{Top})(B) \to \mathscr{H}o(\mathbf{Top})(A)$ is conservative for the same reasons I omitted above. In particular, if A is the set $\mathrm{ob}(B)$ as a discrete category, then we have

$$\mathscr{H}o(\mathbf{Top})(B) \to \mathscr{H}o(\mathbf{Top})(\mathrm{ob}(B)) \simeq \mathscr{H}o(\mathbf{Top})(\mathbb{1})^{\mathrm{ob}(B)}.$$

Thus, if a map $X \to Y$ in $\mathscr{H}o(\mathbf{Top})(B)$ is an isomorphism at all objects, i.e. each $X_a \to Y_a$ is an isomorphism in $\mathscr{D}(\mathbb{1})$, then it is an isomorphism. This property in fact implies the general statement about liberal functors (exercise), but it is the one we generally use in practice, so it is the only one we include in the following definition.

Definition 3.1. A **semiderivator** is a prederivator $\mathscr{D}\colon \mathcal{C}at^{\mathrm{op}} \to \mathcal{CAT}$ with the following properties.

- (Der1) $\mathscr{D}\colon \mathcal{C}at^{\mathrm{op}} \to \mathcal{CAT}$ takes coproducts to products. In particular, $\mathscr{D}(\emptyset)$ is the terminal category.
- (Der2) For any $A \in \mathcal{C}at$, the family of functors $a^*\colon \mathscr{D}(A) \to \mathscr{D}(\mathbb{1})$, as a ranges over the objects of A, is jointly conservative (isomorphism-reflecting).

A semiderivator is **strong** if it satisfies

- (Der5) For any A, the induced functor $\mathscr{D}(A \times \mathbb{2}) \to \mathscr{D}(A)^{\mathbb{2}}$ is full and essentially surjective, where $\mathbb{2} = (0 \to 1)$ is the category with two objects and one nonidentity arrow between them.

(Der3) and (Der4) will show up soon. Note that combined with (Der2), axiom (Der5) implies that the functor in question is weakly smothering. The exact form of (Der5) is negotiable; for instance, Heller assumed a stronger version in which $\mathbb{2}$ is replaced by any finite free category.

We add an extra adjective to indicate (Der5) for historical and technical reasons: (1) it's what other people have done, (2) without it, the notion is 2-categorically algebraic, and (3) not all constructions preserve it.

4. Derivators

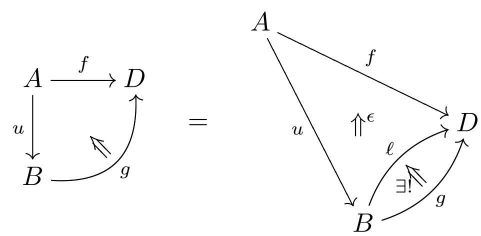
We moved from categories to prederivators because we were hoping the coinverters would behave better. So is the prederivator $\mathscr{H}o(\mathscr{C})$ actually better than the plain homotopy category $\mathscr{C}[\mathscr{W}^{-1}]$? For instance, does it "have limits"?

To answer that question, we need to know what it means for a prederivator to "have limits". There are a bunch of different ways to define what it means for an object of a 2-category to "have limits", but a particularly simple one is with Kan extensions, which are easy to define in any 2-category.

Definition 4.1. A **right Kan extension** of $f : A \to D$ along $u : A \to B$ in any 2-category $\mathcal{K}$ is a morphism $\ell : B \to D$ together with a 2-cell $\epsilon : \ell u \to f$ such that for any $g : B \to D$, composing with ϵ induces a bijection

$$\mathcal{K}(B, D)(g, \ell) \cong \mathcal{K}(A, D)(gu, f).$$

In other words, every 2-cell $gu \to f$ factors uniquely through ϵ:



In yet other words, ℓ has the universal property that a right adjoint to $(- \circ u)$ would have when evaluated at f. Every $f : A \to D$ has such a right Kan extension exactly when $(- \circ u)$ has a right adjoint.

In $\mathcal{C}at$, this reduces exactly to the usual notion of Kan extension. Moreover, in $\mathcal{C}at$ we can identify a *limit* of $f : A \to D$ with its right Kan extension along $A \to \mathbb{1}$. Thus, we might consider defining completeness of an object D in terms of right Kan extensions along maps of the form $A \to 1$.

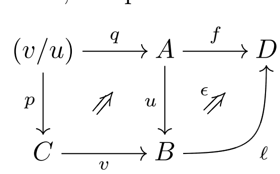
However, it's well-known that when generalizing definitions from familiar examples like **Set** and $\mathcal{C}at$, we often need to replace the terminal object with an arbitrary object. For instance, limits in **Set** can be defined in terms of ordinary elements, which is to say maps out of 1, but to define limits in an arbitrary category we need to consider maps out of arbitrary objects as well ("generalized elements"). Similarly, in $\mathcal{C}at$ we can construct all right Kan extensions out of limits, using the formula for *pointwise Kan extensions*: the right Kan extension of $f : A \to D$ along $u : A \to B$ can be defined by

$$\ell(b) = \lim_{(b \xrightarrow{\beta} u(a)) \in (b/u)} f(a)$$

if this limit exists. Richard said it was a weighted limit, but for **Set**-enriched categories this is equivalent to a limit over a comma category.

In an arbitrary 2-category, we can't just "put together" a morphism ℓ like this, so instead of just Kan extensions to 1, we should ask for general right Kan extensions to exist. Similarly, it no longer follows automatically that this formula is valid even if the limit *does* exist, so we should impose it — otherwise Kan extensions won't behave the way we expect.

Definition 4.2. A right Kan extension in a finitely complete 2-category, as above, is **pointwise** if for any $v : C \to B$, the pasted 2-cell



exhibits ℓv as a right Kan extension of fq along p.

Note that in $\mathcal{C}at$, if we take $C = \mathbb{1}$ and $v = b$ for some $b \in B$, then the assertion that ℓv is a right Kan extension of fq along p says exactly that $\ell(b)$ is the requisite limit. The assertion that it's this *particular* 2-cell says moreover that the universal properties are compatible.

In conclusion, for "completeness" of an object of a 2-category, we should ask that some pointwise Kan extensions exist. We can't ask for *all* of them — we need some size restriction. In the case of prederivators, we have an obvious size restriction to impose: we require A and B to be representables, $y(A)$ and $y(B)$, for ordinary small categories $A, B \in \mathcal{C}at$.

Now things simplify a bit, because the Yoneda lemma says that $\mathcal{PDER}(y(A), \mathscr{D})$ is equivalent to $\mathscr{D}(A)$ for any prederivator $\mathscr{D}$, and so on. Thus, asking for all right Kan extensions along maps of representables just says that $u^* : \mathscr{D}(B) \to \mathscr{D}(A)$ has a right adjoint. We denote this right adjoint by u_*. Moreover, if C is also representable, then the pointwiseness condition reduces to asking that the mate

$$v^* u_* \to p_* p^* u^* v_* \to p_* q^* v^* v_* \to p_* q^*$$

is an isomorphism. The case of non-representable C should follow from the representable one by Yoneda lemma arguments, but we don't need that, so I won't go into it.

We're finally ready for the definition of a derivator!

Definition 4.3. A **derivator** is a semiderivator $\mathscr{D} \colon \mathcal{C}at^{\mathrm{op}} \to \mathcal{CAT}$ which additionally satisfies:

(Der3) Each functor $u^* \colon \mathscr{D}(B) \to \mathscr{D}(A)$ has both a left adjoint $u_!$ and a right adjoint u_*. If $B = \mathbb{1}$, we sometimes write $u_!$ and u_* as colim and lim respectively.

(Der4) For any functors $u \colon A \to C$ and $v \colon B \to C$ in $\mathcal{C}at$, let (u/v) denote their comma category, with projections $p \colon (u/v) \to A$ and $q \colon (u/v) \to B$. Then the canonical mate-transformations

$$q_! p^* \to q_! p^* u^* u_! \to q_! q^* v^* u_! \to v^* u_!$$

and

$$u^* v_* \to p_* p^* u^* v_* \to p_* q^* v^* v_* \to p_* q^*$$

are isomorphisms. (In fact, either is an isomorphism if the other is.)

Combining (Der1) and (Der3), we see that in a derivator, each category $\mathscr{D}(A)$ has actual (small) products and coproducts, since

$$\mathscr{D}(A) \to \mathscr{D}\left(\coprod_X A\right) \simeq \prod_X \mathscr{D}(A)$$

has a right and left adjoint. This makes sense since as we observed above, products and coproducts are usually not a problem homotopically.

I'm not going to *prove* that any of our examples actually are derivators; the ways we currently know to do that require some tools from model category theory or $(\infty, 1)$-category theory, which I don't want to get into. The basic idea is straightforward: we use homotopy limit and colimit constructions to build the functors $u_!$ and u_*. Let's just take it as known that for any "well-behaved" $\mathscr{C}$ and $\mathscr{W}$, the homotopy prederivator $\mathscr{H}o(\mathscr{C})$ is a derivator.

5. Homotopy exactness

There are multiple ways to approach actually *working* with derivators. Indeed, derivators were reinvented independently several times by Grothendieck, Heller, and Franke, and each had their own different approach. The approach I prefer relies heavily on the following notion.

Suppose given any natural transformation in $\mathcal{C}at$ which lives in a square

$$\begin{array}{ccc} D & \xrightarrow{p} & A \\ {\scriptstyle q}\downarrow & \swArrow_{\alpha} & \downarrow{\scriptstyle u} \\ B & \xrightarrow[v]{} & C. \end{array} \tag{5.1}$$

Then by 2-functoriality of $\mathscr{D}$, we have an induced transformation

$$\begin{array}{ccc} \mathscr{D}(C) & \xrightarrow{u^*} & \mathscr{D}(A) \\ {\scriptstyle v^*}\downarrow & \swArrow_{\alpha^*} & \downarrow{\scriptstyle p^*} \\ \mathscr{D}(B) & \xrightarrow[q^*]{} & \mathscr{D}(D). \end{array}$$

Definition 5.2. The square (5.1) is **homotopy exact** if the two mate-transformations

$$q_!p^* \to q_!p^*u^*u_! \xrightarrow{\alpha^*} q_!q^*v^*v_! \to v^*u_! \qquad \text{and}$$
$$u^*v_* \to p_*p^*u^*v_* \xrightarrow{\alpha^*} p_*q^*v^*v_* \to p_*q^*.$$

are isomorphisms in any derivator $\mathscr{D}$.

Thus, (Der4) is exactly the statement that comma squares are homotopy exact. Interestingly, this turns out to imply that a lot of *other* squares are also homotopy exact.

First of all, here's an example where we don't even need (Der4)!

Lemma 5.3. *If $u : A \to B$ is a right adjoint, then the square*

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ \downarrow & & \downarrow \\ \mathbb{1} & \longrightarrow & \mathbb{1} \end{array} \tag{5.4}$$

is homotopy exact.

Proof. If $f \dashv u$, then the given square has a "horizontal" mate

$$\begin{array}{ccc} A & \xleftarrow{f} & B \\ \downarrow & & \downarrow \\ \mathbb{1} & \longleftarrow & \mathbb{1} \end{array}$$

which is also an isomorphism — indeed, an identity — since it is a natural transformation between functors into $\mathbb{1}$. Applying a derivator $\mathscr{D}$, we obtain two isomorphisms which are again horizontal mates of each other:

$$\begin{array}{ccc} \mathscr{D}(A) & \xleftarrow{u^*} & \mathscr{D}(B) \\ \uparrow & & \uparrow \\ \mathbb{1} & \longleftarrow & \mathbb{1} \end{array} \qquad\qquad \begin{array}{ccc} \mathscr{D}(A) & \xrightarrow{f^*} & \mathscr{D}(B) \\ \uparrow & & \uparrow \\ \mathbb{1} & \longleftarrow & \mathbb{1} \end{array}$$

Thus, the vertical mate of the left-hand square is the "total mate" or "conjugate" of the right-hand square, hence also an isomorphism. □

A functor $u : A \to B$ such that (5.4) is homotopy exact is called **homotopy final**. This means that for any $X \in \mathscr{D}(B)$, we have $\operatorname{colim}(u^*X) \cong \operatorname{colim}(X)$: restricting along a homotopy final functor doesn't change the colimit of a diagram.

Now it turns out that limits and colimits, i.e. Kan extensions to $\mathbb{1}$, are often *not* the most convenient kinds of Kan extensions to use in a derivator, because they don't give us a really good handle on the (co)limiting (co)cone. In general, it's better to have an object of $\mathscr{D}(\mathbb{2})$ than a morphism in $\mathscr{D}(\mathbb{1})$ — we can do more with it — and more generally it's better to have objects in some $\mathscr{D}(A)$ than morphisms in some other $D(B)$. (This is the reason why we sometimes need (Der5).) However, the cocone associated to $\operatorname{colim}(X)$ is the unit of the adjunction $\operatorname{colim} = \pi_! \dashv \pi^*$, which consists of morphisms in $\mathscr{D}(A)$.

To remedy this, let $A^{\triangleright}$ denote the category A extended with a new terminal object ∞. Thus, we have a full inclusion $i : A \hookrightarrow A^{\triangleright}$, and we have $A^{\triangleright}(a, \infty) = A^{\triangleright}(\infty, \infty) = \star$ and $A^{\triangleright}(\infty, a) = \emptyset$. Then because ∞ is terminal in $A^{\triangleright}$, there is a comma square

$$\begin{array}{ccc} A & \longrightarrow & A \\ \downarrow & \swarrow & \downarrow{\scriptstyle i} \\ \mathbb{1} & \xrightarrow[\infty]{} & A^{\triangleright} \end{array}$$

Hence, for $X \in \mathscr{D}(A)$, we have $\operatorname{colim}(X) = (i_!(A))_\infty$. Moreover, the diagram $i_!(A)$ contains not only the object $\operatorname{colim}(X)$, but also the input diagram X and the colimiting cocone. To see that it contains X, we observe:

Lemma 5.5. *If $u : A \to B$ is fully faithful, then the following square is homotopy exact:*

$$\begin{array}{ccc} A & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle u} \\ A & \xrightarrow[u]{} & B \end{array}$$

This implies that $u^*u_! = \mathrm{id}$ and $u^*u_* = \mathrm{id}$; thus whenever we Kan extend along a fully faithful functor (such as $i : A \to A^{\triangleright}$), we don't change the "sub-diagram" we started with, we only add new objects to it.

Proof. By (Der2), it suffices to show that the mate $\mathrm{id} \to u^*u_!$ becomes an isomorphism after restricting along $a : \mathbb{1} \to A$, for any $a \in A$. (This is a very important general technique.) Now by functoriality of mates applied to the squares

$$\begin{array}{ccccc} (A/a) & \xrightarrow{p} & A & \longrightarrow & A \\ {\scriptstyle q}\downarrow & \swarrow & \downarrow & & \downarrow{\scriptstyle u} \\ \mathbb{1} & \xrightarrow[a]{} & A & \xrightarrow[u]{} & B, \end{array}$$

the composite of the two mates

$$q_! p^* \operatorname{id}^* \to a^* \operatorname{id}_! \operatorname{id}^* \to a^* u^* u_!$$

is equal to the mate corresponding to the entire pasted rectangle. However, $q_! p^* \to a^*$ is an isomorphism since that is a comma square, so it suffices to show that the pasted rectangle is homotopy exact. But since u is fully faithful, $(A/a) \cong (u/ua)$, so this is also a comma square. □

A particularly important case is pushouts and pullbacks. Let Γ be the category $(\cdot \leftarrow \cdot \to \cdot)$; then $\Gamma^{\triangleright}$ is the square $\square = \mathbb{2} \times \mathbb{2}$:

$$\begin{array}{ccc} (0,0) & \longrightarrow & (0,1) \\ \downarrow & & \downarrow \\ (1,0) & \longrightarrow & (1,1) \end{array}$$

We say a square (i.e. an object of $\mathscr{D}(\square)$) is **cocartesian** if it is in the image of (the fully faithful functor) $(i_\Gamma)_!$, and dually **cartesian** if it is in the image of $(i_{\Gamma^{\mathrm{op}}})_*$.

Lemma 5.6 (Pasting lemma)**.** *Let $\boxminus$ be the category $\mathbb{2} \times \mathbb{3}$, with three inclusions $i_{01}, i_{12}, i_{02} : \square \to \boxminus$ that pick out the left and right squares and the outer rectangle, respectively.*

$$\begin{array}{ccccc} X_{00} & \longrightarrow & X_{01} & \longrightarrow & X_{02} \\ \downarrow & & \downarrow & & \downarrow \\ X_{10} & \longrightarrow & X_{11} & \longrightarrow & X_{12} \end{array}$$

Given $X \in \mathscr{D}(\boxminus)$ such that the left square $(i_{01})^* X$ is cocartesian, then the outer rectangle $(i_{02})^* X$ is cocartesian if and only if the right square $(i_{12})^* X$ is cocartesian.

Proof. Let A be the full subcategory 00-01-01-10 of $\boxminus$, and B the full subcategory 00-01-02-10-11, with inclusions $j : B \to \boxminus$ and $k : A \to B$.

First of all, I claim that if $(i_{01})^* X$ is cocartesian, then $j^* X$ is in the image of $k_!$. Consider the counit $k_! k^* j^* X \to j^* X$. When restricted along k^*, this becomes $k^* k_! k^* j^* X \cong k^* j^* X$, since $k_!$ is fully faithful. Thus it is an isomorphism at all objects of B except possibly 11, so by (Der2) it suffices to check it there. However, when restricted along i_{01}, this is a map between cocartesian squares that becomes an isomorphism in $\mathscr{D}(\Gamma)$. Thus, since $(i_\Gamma)_!$ is fully faithful, this map is also an isomorphism.

Now I claim that if $(i_{01})^* X$ is cocartesian, then $(i_{02})^* X$ being cocartesian and $(i_{12})^* X$ being cocartesian are both equivalent to X being in the image of $j_!$ (and hence equivalent to each other). As before, the counit $j_! j^* X \to X$ is automatically an isomorphism everywhere except possibly 12, by full-faithfulness of $j_!$. Moreover, since $X \cong k_! k^* X$, being in the image of $j_!$ is equivalent to being in the image of $(jk)_!$.

To show that $(i_{02})^* X$ is cocartesian if X is in the image of $(jk)_!$, we show that the following square is homotopy exact:

$$\begin{array}{ccc} \Gamma & \xrightarrow{i_{02}} & A \\ \downarrow & & \downarrow{\scriptstyle jk} \\ \square & \xrightarrow[i_{02}]{} & \boxplus \end{array}$$

It suffices to check this at 12, and there by pasting with a comma square we get

$$\begin{array}{ccccc} \Gamma & \longrightarrow & \Gamma & \xrightarrow{i_{02}} & A \\ \downarrow & \swarrow & \downarrow & & \downarrow{\scriptstyle jk} \\ \mathbb{1} & \xrightarrow[11]{} & \square & \xrightarrow[i_{02}]{} & \boxplus \end{array} \quad = \quad \begin{array}{ccccc} \Gamma & \xrightarrow{i_{02}} & A & \longrightarrow & A \\ \downarrow & & \downarrow & \swarrow & \downarrow{\scriptstyle jk} \\ \mathbb{1} & \longrightarrow & \mathbb{1} & \xrightarrow[12]{} & \boxplus \end{array}$$

But on the right, the right-hand square is a comma square, and the left is homotopy exact since $i_{02} : \Gamma \to A$ is a right adjoint. Homotopy exactness of the right side also implies the converse: if X is in the image of $(jk)_!$, then $(i_{02})^* X$ is cocartesian.

For (i_{12}), we make a similar argument using B, which doesn't even need the assumption that $(i_{01})^* X$ is cocartesian. Here the important fact is that $i_{12} : \Gamma \to B$ is a right adjoint. □

6. Homotopy equivalences

Definition 6.1. A functor $f \colon A \to B$ is a **homotopy equivalence** if the map

$$(\pi_A)_!(\pi_A)^* \cong (\pi_B)_! f_! f^* (\pi_B)^* \to (\pi_B)_!(\pi_B)^*$$

is an isomorphism in any derivator.

Note first that any homotopy final functor, hence in particular any right adjoint, is a homotopy equivalence, since then $(\pi_B)_! f_! f^* \to (\pi_B)_!$ is already an isomorphism.

We could now consider the derivator $\mathscr{H}o(\mathcal{C}at)$ in which we invert the homotopy equivalences. This derivator is "universal" in that it acts on every other derivator: there's a map

$$\odot : \mathscr{H}o(\mathcal{C}at) \times \mathscr{D} \to \mathscr{D}$$

which I won't construct in general, but whose component

$$\odot : \mathscr{H}o(\mathcal{C}at)(\mathbb{1}) \times \mathscr{D}(\mathbb{1}) \to \mathscr{D}(\mathbb{1})$$

is defined by $A \odot X := (\pi_A)_!(\pi_A)^* X$, and this makes $\mathscr{D}$ into a "module" over $\mathscr{H}o(\mathcal{C}at)$.

This is interesting, but $\mathscr{H}o(\mathcal{C}at)$ may still seem somewhat mysterious. Here's the really magical thing about derivators.

Theorem 6.2 (Heller, Cisinski)**.** *$f : A \to B$ is a homotopy equivalence if and only if $A \odot 1 \to B \odot 1$ is an isomorphism in $\mathscr{H}o(\mathbf{Top})$.*

($A \odot 1$ is by definition the homotopy colimit of the constant A-diagram at the one-point space. It's called the **geometric realization** or *classifying space* of the category A.)

In fact, more is true!

Theorem 6.3 (Thomason)**.** *$\mathscr{H}o(\mathcal{C}at)$ is equivalent to $\mathscr{H}o(\mathbf{Top}_w)$.*

Why should this be? What is "universal" about the homotopy theory of spaces? One answer is that the homotopy theory of nice spaces is equivalent to that of ∞-groupoids, but that doesn't completely resolve the mystery, because in defining derivators we didn't build in any notion of "∞-groupoid".

7. Loop spaces

Consider a **pointed object** of $\mathscr{D}$, meaning an object $1 \to X$ of $\mathscr{D}(\mathbb{2})$ whose domain is a terminal object. We define its **loop space** to be the pullback:

(7.1)
$$\begin{array}{ccc} \Omega X & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & X \end{array}$$

(We can obtain this by restricting along $\lrcorner \to \mathbb{2}$ and then right Kan extending.) Note that this is a *coherent* diagram, hence the square should be regarded as commuting up to a specified homotopy. This homotopy assigns to every point of ΩX a path from the basepoint of X to itself, as we expect for the loop space. And universality of this square means that ΩX ought to consist precisely of such paths.

Indeed, it's easy to verify that in topological examples this does what we expect. In algebraic examples like chain complexes, it essentially shifts down by one step, since a chain homotopy from the zero map to itself is essentially just a chain map of degree 1.

Note that here we first see the real power of an honestly homotopy-theoretic notion of limit: it encompasses classical homotopy-theoretic ideas but lets us use almost-ordinary methods of category theory to work with them.

We have defined ΩX to be "the" pullback, in other words we have a specified functor from a subcategory of $\mathscr{D}(\mathbb{2})$ to $\mathscr{D}(\mathbb{1})$. However, in fact, *any* cartesian square

(7.2)
$$\begin{array}{ccc} W & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & X \end{array}$$

induces a canonical isomorphism $W \cong \Omega X$.

It is very important to note that if we restrict a cartesian square (7.2) along the automorphism $\sigma\colon \square \to \square$ which swaps $(0,1)$ and $(1,0)$, we obtain a *different* cartesian square (with the same underlying diagram), and hence a *different* isomorphism $W \cong \Omega X$. The relationship between the two is the following.

Lemma 7.3. *In any derivator, ΩX is a group object in $\mathscr{D}(\mathbb{1})$, and the composite $\Omega X \xrightarrow{\sim} W \xrightarrow{\sim} \Omega X$ of the two isomorphisms arising from a cartesian square (7.2) and its σ-transpose gives the "inversion" morphism of ΩX.*

We generally write this group structure additively, and thus denote this morphism by "-1".

Remark 7.4. This may seem strange, but it is not really a new sort of phenomenon. Already in ordinary category theory, a universal property is not merely a property of an object, but of that object equipped with extra data, and changing the data can give the same object the same universal property in more than one way. For instance, a cartesian product $A \times A$ comes with two projections $\pi_1, \pi_2 \colon A \times A \rightrightarrows A$ exhibiting it as a product of A and A, whereas switching these two projections exhibits the same object as a product of A and A in a different way. In that case, the induced automorphism of $A \times A$ is the symmetry, $(a, b) \mapsto (b, a)$. In the case of suspensions, the "universal property data" consists of a cartesian square (7.1), and transposing the square is analogous to switching the projections.

8. Pointed derivators

It's common in algebraic topology to work in the category of *pointed spaces*, where *everything* is equipped with a chosen basepoint. This puts us in the following situation, which is also the case in algebraic examples:

Definition 8.1. A derivator $\mathscr{D}$ is **pointed** if the category $\mathscr{D}(\mathbb{1})$ has a zero object (an object which is both initial and terminal).

Since $\pi_A^* \colon \mathscr{D}(\mathbb{1}) \to \mathscr{D}(A)$ is both a left and a right adjoint, it preserves zero objects. Hence, in a pointed derivator each category $\mathscr{D}(A)$ also has a zero object.

Since left Kan extension from X to $(0 \to X)$ is fully faithful, and $0 = 1$, it identifies $\mathscr{D}(\mathbb{1})$ with the category of pointed objects as a subcategory of $\mathscr{D}(\mathbb{2})$. In other words, every object is pointed in a unique way. Thus, we have a loop space functor $\mathscr{D}(\mathbb{1}) \to \mathscr{D}(\mathbb{1})$, which can be defined by

$$\mathscr{D}(\mathbb{1}) \xrightarrow{(1,1)_!} \mathscr{D}(\lrcorner) \xrightarrow{(i_\lrcorner)_*} \mathscr{D}(\square) \xrightarrow{(0,0)^*} \mathscr{D}(\mathbb{1}).$$

Now if $\mathscr{D}$ is pointed, so is $\mathscr{D}^{\mathrm{op}}$. The loop space functor of $\mathscr{D}^{\mathrm{op}}$ is called the **suspension** functor of $\mathscr{D}$, and can be defined by the composite

$$\mathscr{D}(\mathbb{1}) \xrightarrow{(0,0)_*} \mathscr{D}(\ulcorner) \xrightarrow{(i_\ulcorner)_!} \mathscr{D}(\square) \xrightarrow{(1,1)^*} \mathscr{D}(\mathbb{1}).$$

Lemma 8.2. *There is an adjunction $\Sigma \dashv \Omega$.*

Proof. Since $i_\ulcorner$ is fully faithful, $(i_\ulcorner)_!$ exhibits $\mathscr{D}(\ulcorner)$ as equivalent to the coreflective subcategory of $\mathscr{D}(\square)$ whose objects are the cocartesian squares (the coreflection being $(i_\ulcorner)^*$). If we write $\mathscr{D}(\ulcorner)_{00}$ and $\mathscr{D}(\square)_{00}$ for the full subcategories of each on the diagrams X such that $X_{0,1}$ and $X_{1,0}$ are zero objects, then both $(i_\ulcorner)^*$ and $(i_\ulcorner)_!$ preserve these subcategories, and so $\mathscr{D}(\ulcorner)_{00}$ is likewise equivalent to the coreflective subcategory of cocartesian squares in $\mathscr{D}(\square)_{00}$. Moreover, $(0,0)_* \colon \mathscr{D}(\mathbb{1}) \to \mathscr{D}(\ulcorner)_{00}$ is an equivalence.

A dual argument using $\mathscr{D}(\lrcorner)_{00}$ shows that $\mathscr{D}(\mathbb{1})$ is also equivalent to the *reflective* subcategory of *cartesian* squares in $\mathscr{D}(\square)_{00}$. Thus, we have a composite adjunction $\mathscr{D}(\mathbb{1}) \rightleftarrows \mathscr{D}(\square)_{00} \rightleftarrows \mathscr{D}(\mathbb{1})$, which is easily verified to be $\Sigma \dashv \Omega$. □

In a pointed derivator we have notions of fiber and cofiber sequence. In general, the fiber of a map into a pointed object should be the pullback of a cospan $Y \to X \leftarrow 1$. In the pointed case, we can obtain this functorially: the **fiber functor** $\mathsf{fib} \colon \mathscr{D}(\mathbb{2}) \to \mathscr{D}(\mathbb{2})$ is the composite

$$\mathscr{D}(\mathbb{2}) \xrightarrow{(-,1)_!} \mathscr{D}(\lrcorner) \xrightarrow{(i_\lrcorner)_*} \mathscr{D}(\square) \xrightarrow{(0,-)^*} \mathscr{D}(\mathbb{2})$$

so that we have a cartesian square

$$\begin{array}{ccc} w & \xrightarrow{\mathsf{fib}(f)} & x \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & y. \end{array}$$

Dually, the fiber functor of $\mathscr{D}^{\mathrm{op}}$ is the **cofiber functor** of $\mathscr{D}$, and we have an adjunction $\mathsf{cof} \dashv \mathsf{fib}$.

Remark 8.3. In a *strong* pointed derivator, every morphism in $\mathscr{D}(\mathbb{1})$ underlies some object of $\mathscr{D}(\mathbb{2})$. Thus, we can construct "the" fiber or cofiber of any morphism in $\mathscr{D}(\mathbb{1})$ by first lifting it to an object of $\mathscr{D}(\mathbb{2})$. Since weakly smothering functors reflect the isomorphism relation, the result is independent of the chosen lift, up to *non-unique isomorphism*.

In a pointed derivator $\mathscr{D}$, we define a **fiber sequence** to be a coherent diagram of shape $\boxbar = \mathbb{2} \times \mathbb{3}$ in which both squares are cartesian and whose $(0,2)$- and $(1,0)$-entries are zero objects:

$$\begin{array}{ccccc} w & \xrightarrow{f} & x & \longrightarrow & 0 \\ \downarrow & & \downarrow g & & \downarrow \\ 0 & \longrightarrow & y & \xrightarrow[h]{} & z \end{array}$$

Suitable combinations of Kan extensions give a functorial construction of fiber sequences $\mathscr{D}(\mathbb{2}) \to \mathscr{D}(\boxbar)$, which induces an equivalence onto the full subcategory of $\mathscr{D}(\boxbar)$ spanned by the fiber sequence.

Recall that ι_{jk} denotes the functor $\square \to \boxbar$ induced by the identity of $\mathbb{2}$ on the first factor and the functor $\mathbb{2} \to \mathbb{3}$ on the second factor which sends 0 to j and 1 to k. Then a fiber sequence is an $X \in \mathscr{D}(\boxbar)$ such that $X_{(0,2)}$ and $X_{(1,0)}$ are zero objects and $\iota_{01}^* X$ and $\iota_{12}^* X$ are cartesian. By the pasting lemma, $\iota_{02}^* X$ is also cartesian, and therefore induces an isomorphism $w \cong \Omega x$.

We can thus continue a fiber sequence indefinitely to the left:

$$\cdots \to \Omega^2 z \to \Omega x \to \Omega y \to \Omega z \to x \to y \to z$$

9. Stable derivators

In an algebraic example of *unbounded* chain complexes, we saw that Ω was the "shift" functor. This is actually an *equivalence*, since we can also shift the other direction.

Theorem 9.1. *For a pointed derivator $\mathscr{D}$, the following are equivalent:*

(i) The adjunction $\Sigma \dashv \Omega$ is an equivalence.
(ii) The adjunction $\mathsf{cof} \dashv \mathsf{fib}$ is an equivalence.
(iii) A square $X \in \mathscr{D}(\square)$ is cartesian if and only if it is cocartesian.

*Such a derivator is called **stable**.*

Stable derivators (or model categories, or $(\infty,1)$-categories) are the homotopy-theoretic version of *abelian categories*: they are the place where we do homotopy-theoretic commutative algebra and homological algebra. As a first evidence of this, we have:

Theorem 9.2. *A stable derivator is preadditive, i.e. the map $X \sqcup Y \to X \times Y$ in $\mathscr{D}(\mathbb{1})$ is an isomorphism.*

Proof. We left extend from

$$\begin{array}{cc} & X \\ Y & \end{array} \qquad \text{to} \qquad \begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \sqcup Y. \end{array}$$

We identify the given objects by comma categories. Now we extend by zero to

$$\begin{array}{ccccc} 0 & \longrightarrow & X & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ Y & \longrightarrow & X \sqcup Y. & & \\ \downarrow & & & & \\ 0 & & & & \end{array}$$

and left extend again to

$$\begin{array}{ccccc} 0 & \longrightarrow & X & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & X \sqcup Y & \longrightarrow & Y \\ \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X & & \end{array} \qquad \text{and then} \qquad \begin{array}{ccccc} 0 & \longrightarrow & X & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & X \sqcup Y & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X & \longrightarrow & 0. \end{array}$$

Comma categories and homotopy finality show that all squares are cocartesian, hence so are all rectangles, and thus we can identify the labeled objects as shown. But now the lower-right square is also cartesian, exhibiting $X \sqcup Y$ as $X \times Y$. □

We write $X \oplus Y$ for the common value and call it a **direct sum** or **biproduct**. We can then add morphisms $f, g : X \to Y$ in the usual way:

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla} Y.$$

Thus $\mathscr{D}$ is enriched over abelian monoids.

In fact, $\mathscr{D}(\mathbb{1})$ is not just preadditive but *additive* — this operation makes the homsets abelian groups, not just abelian monoids. We already mentioned the inversion map. Indeed, we could also derive the above theorem from the fact that loop space objects are groups and double loop spaces are abelian groups, since in a stable derivator everything is a double loop space, $X \cong \Omega^2\Sigma^2 X$.

A stable derivator is actually *better* than an abelian category, essentially because its colimits don't lose information. In an abelian category, if you take the quotient by a subobject, then you can recover the subobject as the kernel of the quotient map. But if you take the quotient of a map that isn't injective, then you can't recover the map itself, only its image. By contrast, in a stable derivator, from the quotient (= cofiber) of *any* map we can recover that map as the fiber of the quotient map, using bicartesianness:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z. \end{array}$$

The most classical way to deal with stability is via the following.

Theorem 9.3. *If $\mathscr{D}$ is stable, then $\mathscr{D}(\mathbb{1})$ is a* ***triangulated category*** *in the sense of Verdier.*

A triangulated category is equipped with an autoequivalence Σ and a collection of composable strings of morphisms of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

called *distinguished triangles* satisfying certain axioms. In a stable derivator, we take these to underlie the cofiber sequences, and prove the axioms.

The structure of a stable derivator is much better behaved than that of a triangulated category, since everything has a universal property. Triangulated categories have the problem that things are asserted to exist with some property, but are not *characterized* by that property, and other random things can also exist with the same property.

10. Descent

Two of the most important strains of ordinary category theory are the theory of *abelian* and similar categories — generalizations of abelian groups — and the theory of *toposes* and similar categories — generalizations of sets. Stable derivators are the homotopy version of abelian categories, which we've seen are better-behaved. Let's look for a homotopy version of toposes.

One of the most important properties of a topos is its exactness properties. The first one is this:

Definition 10.1. An ordinary category with finite limits is called **lextensive** if it has (finite) coproducts which are *stable* (i.e. preserved by pullback) and *disjoint*, i.e. the coproduct injections are monic and we have a pullback square

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X+Y \end{array}$$

This may look a little *ad hoc*, but it's equivalent to the following: given a map of cocones under a discrete category:

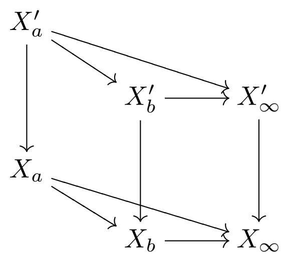
$$\begin{array}{ccccc} X' & \longrightarrow & X'+Y' & \longleftarrow & Y' \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & X+Y & \longleftarrow & Y \end{array}$$

whose codomain is a coproduct cocone, then the squares are pullbacks iff the domain is also a coproduct cocone. It's also equivalent to asking that the pseudofunctor

$$\mathscr{C}^{\mathrm{op}} \longrightarrow \mathcal{CAT}$$
$$X \mapsto \mathscr{C}/X$$

takes coproducts to products.

This is quite pretty, and should make you itch to generalize it from coproducts to more general colimits. There's one modification we have to make in order for it to be vaguely sensible: if we have a map $a \to b$ in the diagram category A, then our map of cocones would include data like



and if the squares into the cocone vertices are to be pullbacks, then the pasting lemma would imply that the other square must be a pullback. So we should include this in the hypotheses (it being vacuous when A is discrete).

Recall $i : A \to A^{\triangleright}$ and $\mathbb{2} = (0 \to 1)$. A diagram $X \in \mathscr{D}(A \times \mathbb{2})$ is said to be **equifibered** or **cartesian** if for every $j : \mathbb{2} \to A$, the induced square $(j \times \mathrm{id})^* X$ is cartesian.

Definition 10.2. A derivator $\mathscr{D}$ has **descent for A-colimits** if for any $X \in \mathscr{D}(A^{\triangleright} \times \mathbb{2})$ such that $X_1 \in \mathscr{D}(A^{\triangleright})$ is colimiting and $(i \times \mathrm{id})^* X$ is equifibered, the following are equivalent:

(i) X_0 is colimiting.
(ii) X is equifibered.

Unfortunately, even **Set** doesn't have descent for non-discrete colimits! Consider pushouts; here is an equifibered diagram over $\ulcorner$:

$$\begin{array}{ccccc} 2 & \xleftarrow{[\mathrm{id},\mathrm{id}]} & 2+2 & \xrightarrow{[\mathrm{id},s]} & 2 \\ \downarrow & & \downarrow & & \downarrow \\ 1 & \longleftarrow & 1+1 & \longrightarrow & 1 \end{array}$$

where $s : 2 \to 2$ is the switch automorphism. But the pushout of both rows is 1, while the resulting squares

$$\begin{array}{ccc} 2 & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & 1 \end{array}$$

are obviously not pullbacks.

Fortunately, there are better replacements. Clearly the problem is that the pushouts are *losing information*, just like the quotients classical abelian categories. Thus, using homotopy colimits instead, we could hope to remedy the problem. In fact, the classical homotopy derivator of spaces with weak homotopy equivalence has descent for *all* colimits!

10.1. **The loop space of the circle.** The following proof is inspired by homotopy type theory.

Let $P = (a \rightrightarrows b)$ be the free-living parallel pair. In a derivator satisfying stability and descent for P-colimits, let S^1 denote the colimit of the constant P-diagram $(\pi_P)^*(1)$ at the terminal object. We will show that $\Omega(S^1)$ is "$\mathbb{Z}$", or more precisely is the coproduct of countably many copies of 1.

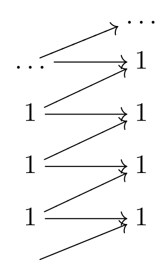
Let $F : P \times \mathbb{2} \to \mathcal{C}at$ be defined by $F(a, 0) = F(b, 0) = \mathbb{Z}$ and $F(x, 1) = 1$, with the images of the two arrows $(a, 0) \rightrightarrows (b, 0)$ being the identity and the successor function respectively. Let Q be the Grothendieck construction of F, with induced discrete opfibration $p : Q \to P \times \mathbb{2}$, and consider $X = (i \times \mathrm{id})_! p_! (\pi_Q)^*(1)$ where $i \times \mathrm{id} : P \times \mathbb{2} \to P^{\rhd} \times \mathbb{2}$ is the inclusion. Since p is a discrete opfibration, $p_!(\pi_Q)^*(1)$ looks like

$$\begin{array}{ccc} \coprod_{\mathbb{Z}} 1 & \rightrightarrows & \coprod_{\mathbb{Z}} 1 \\ \downarrow & & \downarrow \\ 1 & \rightrightarrows & 1 \end{array}$$

and its restrictions to $P \times \{0\}$ and $P \times \{1\}$ can be computed by first restricting $(\pi_Q)^*(1)$ to the corresponding subcategories of Q. Moreover, since the left Kan extension to $P^{\rhd} \times \mathbb{2}$ is performed levelwise, X looks like

$$\begin{array}{ccccc} \coprod_{\mathbb{Z}} 1 & \rightrightarrows & \coprod_{\mathbb{Z}} 1 & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ 1 & \rightrightarrows & 1 & \longrightarrow & S^1 \end{array}$$

where Y is the colimit of $\coprod_{\mathbb{Z}} 1 \rightrightarrows \coprod_{\mathbb{Z}} 1$. This can equivalently be described as the colimit of a constant diagram on 1 over the full subcategory $Q_0 \subseteq Q$, which looks like this:



This clearly has a contractible nerve, so $Y = 1$. Thus, to show that $\Omega S^1 \cong \coprod_{\mathbb{Z}} 1$ it will suffice to show that X is equifibered. By descent, for this it will suffice to show that the restriction of X to $P \times \mathbb{2}$ is equifibered. But in this case all the horizontal morphisms are isomorphisms, so all squares are automatically cartesian.