2-DIMENSIONAL AND MONOIDAL CATEGORIES

RICHARD GARNER
NOTES BY EMILY RIEHL

CONTENTS

1. 2-CATEGORY THEORY

2-category theory is a special case of enriched category theory, but there are some features particular to that case.

2-category theory.

Definition. A **2-category** $\mathcal{K}$ is given by

- a set of object ob$\mathcal{K}$;
- for each pari of objects $X, Y \in \mathcal{K}$ a **hom-category** $\mathcal{K}(X, Y)$
  - write objects $f \in \mathcal{K}(X, Y)$ as $f : X \to Y$ (1-cells)
  - morphisms $\alpha : f \to g \in \mathcal{K}(X, Y)$ as $\alpha : X \underset{g}{\overset{f}{\Downarrow}} Y$ (2-cells)

  We have identity 2-cells and vertical composition of 2-cells.
- composition functor $\mathcal{K}(Y, Z) \times \mathcal{K}(X, Y) \to \mathcal{K}(X, Z)$ which defines composition of 1-cells and vertical composition of 2-cells. In particular, we have whiskering:

$$Y \underset{g}{\overset{g}{\Downarrow 1_g}} Z \ , \ X \underset{f'}{\overset{f}{\Downarrow \beta}} Y \ \mapsto \ X \underset{gf'}{\overset{gf}{\Downarrow g\beta}} Z$$

- nullary composition: $1 \to \mathcal{K}(X, X)$ denoted by $* \mapsto 1_X : X \to X$. Finally, we have
- axioms assuring that every way of composition 0-,1-, and 2-cells yields the same result.

Everything we'll say about 2-categories has analogs in the bicategorical world.

Example.

- **Cat**—categories, functors, and natural transformations

Date: 2nd July 2013.
Corrections to `eriehl@math.harvard.edu`.

- **MonCat**$_{s(p,\ell,c)}$—monoidal categories, strict (strong, lax, colax) monoidal functors, and monoidal natural transformations
- **Lex**—categories with finite limits, limit preserving functors, and natural transformations
- a one-object 2-category is a strict monoidal category

The project of formal category theory is to generalize the basic results of category theory from **Cat** to other 2-categories.

Functor 2-categories. For categories there's just one notion of functor category. For 2-categories there are 16 sensible combinations of what we might want for a functor 2-category.

A **2-functor** $\mathcal{K} \xrightarrow{F} \mathcal{L}$ is given by assignations on 0-, 1-, and 2-cells which preserve all forms of composition strictly.

In the 2-categorical case, any equality in the 1-categorical place can be replaced by either an invertible or non-invertible 2-cell. These 2-cells provide additional data, which is then required to be "coherent." I'll give one example of what this means and then not worry about it.[1]

A **pseudofunctor** $\mathcal{K} \xrightarrow{F} \mathcal{L}$ is given by assignations on 0-, 1-, and 2-cells plus:

- for each $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{K}$, an invertible 2-cell $F_{f,g} \colon Fg \cdot Ff \Rightarrow F(g \cdot f) \colon FA \to FC$ in $\mathcal{L}$
- for each $A \xrightarrow{1_A} A$ in $\mathcal{K}$, an invertible 2-cell $F_A \colon 1_{FA} \Rightarrow F(1_A) \colon FA \to FA$ in $\mathcal{L}$

satisfying axioms:

-
$$\begin{array}{ccc} Fh \cdot Fg \cdot Ff & \xRightarrow{Fh \cdot F_{f,g}} & Fh \cdot F(g \cdot f) \\ {\scriptstyle F_{g,h} \cdot Ff} \Downarrow & & \Downarrow {\scriptstyle F_{gf,h}} \\ F(h \cdot g) \cdot Ff & \xRightarrow[F_{f,hg}]{} & F(h \cdot g \cdot f) \end{array}$$
- two other axioms involving the unit
- other axioms involving 2-cells in $\mathcal{K}$.

Note there are certain sorts of composition that are still preserved strictly, e.g. vertical composition of 2-cells. The reason is you can't replace this sort of equality by an invertible cell because there are no cells in higher dimensions. A reference is [KS].

Example.

- a pseudofunctor between one-object 2-categories (strict monoidal categories) is a strong monoidal functor
- Let C be a category with pullbacks. There's a pseudofunctor $C^{\mathrm{op}} \to \mathbf{CAT}$ given by $X \mapsto C/X$ and $f \colon X \to Y \mapsto C/Y \xrightarrow{f^*} C/X$.

A **lax functor** $\mathcal{K} \xrightarrow{F} \mathcal{L}$ is the same data and axioms as a pseudofunctor except the 2-cells F_A and $F_{f,g}$ are not necessarily invertible. A **oplax functor** is obtained by orienting F_A and $F_{f,g}$ in the opposite direction. The pseudofunctors are contained in the intersection of the lax and the oplax things.[2]

---

[1] You just kind of sit and stare at it and write down some obvious things and hope you have enough, and usually you have.

[2] Richard said *are* the intersection, then Steve objected.

Between each of these kinds of functor, we have various kinds of transformation. We'll concentrate on the case of 2-functors for simplicity. Given $F, G\colon \mathcal{K} \rightrightarrows \mathcal{L}$, a **2-natural transformation** $\alpha\colon F \Rightarrow G$ is given by:

- components $\alpha_X\colon FX \to GX$ for each $X \in \mathcal{K}$ satisfying the usual naturality condition and also, for $X \underset{g}{\overset{f}{\Downarrow\gamma}} Y$ in $\mathcal{K}$, we have

$$FX \underset{Fg}{\overset{Ff}{\Downarrow F\gamma}} FY \xrightarrow{\alpha_Y} GY \;=\; FX \xrightarrow{\alpha_X} GX \underset{Gg}{\overset{Gf}{\Downarrow G\gamma}} GY$$

A **pseudo natural transformation** $\alpha\colon F \Rightarrow G$ is given by components $\alpha_X\colon FX \to GX$ in $\mathcal{L}$ for all $X \in \mathcal{K}$ plus invertible 2-cell components

$$\begin{array}{ccc} FX & \xrightarrow{\alpha_X} & GX \\ {\scriptstyle Ff}\downarrow & \Downarrow\alpha_f & \downarrow{\scriptstyle Gf} \\ FY & \xrightarrow[\alpha_Y]{} & GY \end{array}$$

in $\mathcal{L}$ for each map $f\colon X \to Y$ in $\mathcal{K}$. These again have to satisfy some axioms about composition and identities.

A **lax natural transformation** is as before—now the α_f is not necessarily invertible—as is an **oplax natural transformation**—now the α_f is not necessarily invertible and reversed in direction.

Because 2-categories have an extra dimension there is an extra dimension of maps between them: modifications. Given, say, pseudonatural transformations $\alpha, \beta\colon F \Rightarrow G\colon \mathcal{K} \to \mathcal{L}$ a modification $\Gamma\colon \alpha \Rrightarrow \beta$ is given by components $\Gamma_X\colon \alpha_X \Rightarrow \beta_X\colon FX \to GX$ in $\mathcal{L}$ for all $X \in \mathcal{K}$ satisfying axioms:

- $$\begin{array}{ccc} Gf\cdot\alpha_X & \xRightarrow{\alpha_f} & \alpha_Y\cdot Ff \\ {\scriptstyle Gf\cdot\Gamma_X}\Downarrow & & \Downarrow{\scriptstyle \Gamma_Y\cdot Ff} \\ Gf\cdot\beta_X & \xRightarrow[\beta_f]{} & \beta_Y\cdot Ff \end{array}$$

  commutes for all $f\colon X \to Y$ in $\mathcal{K}$.

Now if $\mathcal{K}$ and $\mathcal{L}$ are 2-categories we have various kinds of functor 2-category:

- objects are 2-, pseudo-, lax-, or oplax functors
- 1-cells are 2-, pseudo, lax, or oplax natural transformations
- 2-cells are modifications

Remark. An important case is functor categories into **Cat**. If $\mathcal{K}$ is a locally small 2-category we have a Yoneda embedding $\mathcal{K} \to [\mathcal{K}^{\mathrm{op}}, \mathbf{Cat}]$, where we use square brackets to denote the strictest case: 2-functors, 2-natural transformations, and modifications.

Relations between these functor categories[3]**.** Fix $\mathcal{K}$ and $\mathcal{L}$. Write $\mathbf{Lax}(\mathcal{K}, \mathcal{L})_s$ for the 2-category of lax functors, strict 2-natural transformations, and modifications. The inclusion $J\colon [\mathcal{K}, \mathcal{L}] \to \mathbf{Lax}(\mathcal{K}, \mathcal{L})_s$ has both left and right 2-adjoints if $\mathcal{K}$ is small and $\mathcal{L}$ is

[3]Maybe this is the first thing that I'll say that doesn't involve just defining reams of stuff.

complete and cocomplete: In fact, we can identify $\mathbf{Lax}(\mathcal{K},\mathcal{L})_s$ with $[\mathcal{K}^\dagger,\mathcal{L}]$ (an isomorphism of 2-categories) for another 2-category $\mathcal{K}^\dagger$. So if $\mathcal{K}$ is small and $\mathcal{L}$ is complete (resp. cocomplete) then J has a right (resp. left) 2-adjoint.

What is $\mathcal{K}^\dagger$? Objects are those of $\mathcal{K}$. 1-cells are strings of composable 1-cells of $\mathcal{K}$. A 2-cell α from $X \xrightarrow{f_1} \cdots \xrightarrow{f_n} Y$ to $X \xrightarrow{g_1} \cdots \xrightarrow{g_m} Y$ is given by an order preserving map $\{1,\ldots,n\} \xrightarrow{\varphi} \{1,\ldots,m\}$ and 2-cells $\alpha_1,\ldots,\alpha_m$ where

$$\alpha_i \colon \circ_{j\in\varphi^{-1}(i)} f_j \Rightarrow g_i.$$

Note in order for this 2-cell to exist these 1-cells must have the same source and target, which is an additional condition.

Exercise. A 2-functor $\mathcal{K}^\dagger \to \mathcal{L}$ is a lax functor $\mathcal{K} \to \mathcal{L}$.

Monads in a 2-category. The case of interest of us for this talk will be $\mathcal{K} = \mathbb{1}$.

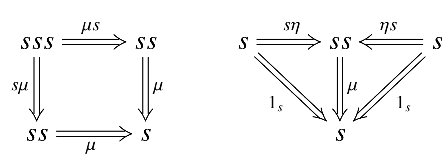
Definition. A **monad** in a 2-category $\mathcal{L}$ is a lax functor $\mathbb{1} \xrightarrow{F} \mathcal{L}$.

What is this? Writing $*$ for the single object, we have $* \mapsto F(*) = A$, $* \xrightarrow{1_*} * \mapsto F(1) = s\colon A \to A$, and $* \underset{1_*}{\overset{1_*}{\Downarrow 1_{1_*}}} * \ \mapsto 1_s\colon s \Rightarrow s\colon A \to A$ (by one of the axioms for lax functors).

Plus

- $1_{F*} \Rightarrow F(1_*)\colon F(*) \Rightarrow F(*)$ i.e., $\eta\colon 1_A \Rightarrow s\colon A \to A$ (preservation of nullary composition)
- $F(1_*)\cdot F(1_*) \Rightarrow F(1_*\cdot 1_*)\colon F(*) \to F(*)$ i.e., $\mu\colon s\cdot s \Rightarrow s\colon A \to A$.

Plus axioms



What is $\mathbb{1}^\dagger$?

- single object *
- 1-cells $* \to *$ are natural numbers including 0 (the empty string)
- 2-cells $n \Rightarrow m$ are order preserving maps $\{1,\ldots,n\} \to \{1,\ldots,m\}$

i.e., $\mathbb{1}^\dagger(*,*) = \mathbb{\Delta}_+$. There is the topologist's delta, which contains finite non-empty ordinals and order preserving maps. This is the algebraist's delta, which contains all finite ordinals (including the empty ordinal) and order preserving maps.[4]

As a one-object 2-category, this makes $\mathbb{\Delta}_+$ a strict monoidal category: the monoidal structure is addition of natural numbers. (This is where the algebraist's delta differs from the topologist's delta, which is not a monoidal category because it has no unit.)

We write $\mathbb{1}^\dagger$ as $\Sigma\mathbb{\Delta}_+$ and so have that 2-functors $\Sigma\mathbb{\Delta}_+ \to \mathcal{L}$ correspond to monads in $\mathcal{L}$.

[4]There is a further confusion: The objects of the topologist's delta are the natural numbers. The objects of the algebraist's delta are also the natural numbers, but these are not the same natural numbers.

Adjunctions in a 2-category. We had a mildly slick way of defining monads in a 2-category. For adjunctions the best way is just to do it. An **adjunction** in a 2-category $\mathcal{L}$ is given by $f\colon A \leftrightarrows B\colon g$ objects and 1-cells and 2-cells $\eta\colon 1_A \Rightarrow gf\colon A \to A$ and $\epsilon\colon fg \Rightarrow 1_B\colon B \to B$ such that

$$\begin{array}{ccc} f & \overset{f\eta}{\Longrightarrow} & fgf \\ & {\scriptstyle 1_f}\searrow\!\!\searrow & \Downarrow{\scriptstyle \epsilon f} \\ & & f \end{array} \qquad \begin{array}{ccc} g & \overset{\eta g}{\Longrightarrow} & gfg \\ & {\scriptstyle 1_g}\searrow\!\!\searrow & \Downarrow{\scriptstyle g\epsilon} \\ & & g \end{array}$$

Mates. Any 2-functor preserves adjunctions. In particular, if $f\colon A \leftrightarrows B\colon g$ is an adjunction in $\mathcal{L}$, we can apply the hom-fucntor $\mathcal{L}(X,-)\colon \mathcal{L} \to \mathbf{Cat}$ for any $X \in \mathcal{L}$ to get an adjunction

$$\mathcal{L}(X,A) \underset{f\cdot -}{\overset{g\cdot -}{\underset{\top}{\leftrightarrows}}} \mathcal{L}(X,B)$$

in **Cat**. So we have natural isomorphisms between $\mathcal{L}(X,B)(f\cdot h,k) \cong \mathcal{L}(X,A)(h,g\cdot k)$.

We can also apply a contravariant hom-functor $\mathcal{L}(-,X)\colon \mathcal{L}^{\mathrm{op}} \to \mathbf{Cat}$ to get an adjunction

$$\mathcal{L}(A,X) \underset{-\cdot f}{\overset{-\cdot g}{\underset{\bot}{\rightleftarrows}}} \mathcal{L}(B,X)$$

So we have isomorphisms of hom-categories $\mathcal{L}(A,X)(h\cdot g,k) \cong \mathcal{L}(B,X)(h,k\cdot f)$.[5]

This is some part of the thing that's called mates. Mates, precisely: given

$$\begin{array}{ccc} A & \underset{g_1}{\overset{f_1}{\underset{\bot}{\rightleftarrows}}} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ C & \underset{g_2}{\overset{f_2}{\underset{\bot}{\rightleftarrows}}} & D \end{array}$$

then $\mathcal{L}(A,D)(f_2h,kf_1) \cong \mathcal{L}(B,C)(hg_1,g_2k)$. I.e., 2-cells α correspond to 2-cells $\overline{\alpha}$:

$$\begin{array}{ccc} A & \xrightarrow{f_1} & B \\ {\scriptstyle h}\downarrow & \Rightarrow\alpha & \downarrow{\scriptstyle k} \\ C & \xrightarrow{f_2} & D \end{array} \qquad \begin{array}{ccc} A & \xleftarrow[g_1]{} & B \\ {\scriptstyle h}\downarrow & \Leftarrow\overline{\alpha} & \downarrow{\scriptstyle k} \\ C & \xleftarrow[g_2]{} & D \end{array}$$

Proof: $f_2h \Rightarrow kf_1$ corresponds to $h \Rightarrow g_2kf_1$ corresponds to $hg_1 \Rightarrow g_2k$.

The free adjunction. We had a 2-category classifying monads in the sense that 2-functors with this domain corresponded to 2-adjunctions. We now want to do the same thing for adjunctions. The reference is a four page paper [SS].

There's a 2-category **Adj** so that 2-functors $\mathbf{Adj} \to \mathcal{L}$ are adjunctions in $\mathcal{L}$. It has two objects A and B and

- $\mathbf{Adj}(A,A) = \mathbb{\Delta}_+$
- $\mathbf{Adj}(B,B) = \mathbb{\Delta}_+^{\mathrm{op}}$

[5] We can move an f on the right on the right to a g on the right on the left.

- **Adj**(B, A) has objects $\{u, ufu, ufufu, \ldots\}$ which we identify with natural numbers $\{1, 2, \ldots\}$. Morphisms $n \to m$ are order preserving maps $\{1, \ldots, n\} \to \{1, \ldots, m\}$ sending 1 to 1. E.g., $4 \to 3$ given by $1, 2 \mapsto 1$; $3, 4 \mapsto 2$ is the thing we might label

$$ufufufu \xRightarrow{u\epsilon fu\epsilon} ufu \xRightarrow{\eta ufu} ufufu$$

- **Adj**$(A, B) =$ **Adj**$(B, A)^{\mathrm{op}}$.

Limits and colimits in a 2-category. 2-(co)limits are a special case of enriched (co)limits. I'm going to start by just listing a bunch of examples.

Example (conical limits and colimits)**.** Given a diagram $D\colon \mathcal{I} \to \mathcal{L}$, $\mathcal{I}$ a 1-category and $\mathcal{L}$ a 2-category, a **2-limit** for D is a limit for D in the underlying 1-category of $\mathcal{L}$ which is preserved by each representable $\mathcal{L}(X, -)\colon \mathcal{L} \to$ **Cat**, i.e.,

$$\mathcal{L}(X, \lim D) \cong \lim \mathcal{L}(X, D)$$

as categories. Similarly, for colimits using the contravariant representables.

Example (cotensors and tensors[6])**.** If C is a small category and $X \in \mathcal{L}$ then the **cotensor** of X by C, written $C \pitchfork X$ is characterized by a 2-natural isomorphism

$$\mathcal{L}(Y, C \pitchfork X) \cong \mathcal{L}(Y, X)^C.$$

In particular, taking $C = \mathbb{2}$ then $\mathbb{2} \pitchfork X$ is an object of $\mathcal{L}$ equipped with

$$\mathbb{2} \pitchfork X \underset{}{\overset{\Downarrow\gamma}{\rightrightarrows}} X$$

such that every $C \underset{g}{\overset{f}{\rightrightarrows}} X$ (with 2-cell $\Downarrow\alpha$) factors uniquely as

$$C \xrightarrow{\langle\alpha\rangle} \mathbb{2} \pitchfork X \overset{\Downarrow\gamma}{\rightrightarrows} X$$

and there's a further 2-dimensional aspect of this universal property.

The **tensor** of X by C, $C \otimes X$ satisfies

$$\mathcal{L}(C \otimes X, Y) \cong \mathcal{L}(X, Y)^C$$

2-natural in Y.

All other limit and colimit types can be constructed from these two examples in the sense that all ordinary limits can be constructed from products and equalizers. This is not to say that if a particular limit exists then it had to be constructed in this way, from conical limits and cotensors. Let's look at some other specific limit and colimit types.

Example. Given $A \underset{g}{\overset{f}{\rightrightarrows}} B$ the **inserter** of f and g is the universal object $C \xrightarrow{i} A$ for which there is a 2-cell

$$C \underset{gi}{\overset{fi}{\rightrightarrows}} B \quad (\Downarrow\gamma)$$

[6]The limit one is the cotensor and the colimit is called a tensor.

In this context, universal means if given $C' \underset{gi'}{\overset{fi'}{\Downarrow\gamma'}} B$ then there is a unique $C' \to C$ so that the restriction of γ along this map is γ—and there is a 2-dimensional aspect of this universal property too. In **Cat**, the inserter of f and g is the category whose objects are pairs $(a \in A, \gamma_a : fa \to ga)$.

Example. Given a cospan $A \xrightarrow{f} C \xleftarrow{g} B$ the **comma object** is the universal data

$$\begin{array}{ccc} D & \xrightarrow{h} & B \\ {\scriptstyle k}\downarrow & \Rightarrow & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

Typical notation is $D = f \downarrow g$. In **Cat**, D has objects being triples $(a \in A, b \in B, \gamma : fa \to gb \in C)$.

There are dual colimit notions for both of these.

Example. Given $A \underset{g}{\overset{f}{\Downarrow\gamma}} B$ the **inverter** is the universal $A' \xrightarrow{h} A$ such that γh is an invertible 2-cell. In **Cat** the inverter is the full subcategory of A on those $a \in A$ so that γ_a is invertible.

The dual notion is called a **coinverter**. The coinverter of γ is the universal $B \xrightarrow{q} B'$ so that $q\gamma$ is invertible. In **Cat**, 2-limits are easy to describe, while colimits in **Cat**, like colimits in **Set**, have to be described by some inductive process. The coinverter in **Cat** is defined by $B' = B[\Sigma^{-1}]$ where $\Sigma = \{\gamma_a : fa \to ga, \forall a \in A\}$.

We've claimed that 2-colimits can be built from tensors and conical colimits. E.g., we can construct the coinverter of γ from tensors and pushouts: the 2-cell γ corresponds to the 1-cell $\mathbb{2} \otimes A \xrightarrow{\langle\gamma\rangle} B$. Writing $\mathbb{I}$ for the walking isomorphism, the pushout

$$\begin{array}{ccc} \mathbb{2} \otimes A & \xrightarrow{\langle\gamma\rangle} & B \\ \downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{I} \otimes A & \longrightarrow & B' \end{array}$$

defines the coinverter of the 2-cell γ.

Definition. A 2-category is **complete** if it admits conical limits and cotensors and **cocomplete** if it admits conical colimits and tensors. A 2-functor is **continuous** or **cocontinuous** if it preserves these.

Definition. Let $\mathcal{L}$ be a cocomplete 2-category. Let $\mathcal{I}$ be a small 2-category and $D : \mathcal{I} \to \mathcal{L}$ a 2-functor. We define the **weighted colimit 2-functor**

$$(-) \star D : [\mathcal{I}^{\mathrm{op}}, \mathbf{Cat}] \to \mathcal{L}$$

by two conditions:

(i) $(-) \star D$ is cocontinuous
(ii) $\mathcal{I} \xrightarrow{y} [\mathcal{I}^{\mathrm{op}}, \mathbf{Cat}] \xrightarrow{(-)\star D} \mathcal{L} \cong D$

Why do these define a 2-functor. Given a small 2-category $\mathcal{I}$, $[\mathcal{I}^{\mathrm{op}}, \mathbf{Cat}]$ is the free cocompletion of $\mathcal{I}$ under 2-dimensional colimits. We call the value $\varphi \star D$ of this 2-functor at $\varphi \in [\mathcal{I}^{\mathrm{op}}, \mathbf{Cat}]$ the **weighted colimit** of D by φ.

Let's see, e.g., how to express coinverters as a weighted colimit. Take $\mathcal{I} = \; 0 \underset{g}{\overset{f}{\rightrightarrows}} 1$ (with a 2-cell $\Downarrow\gamma$ from f to g).

Then $D\colon \mathcal{I} \to \mathcal{L}$ picks out a 2-cell in $\mathcal{L}$. Now $\mathcal{I}(-,0) \star D = D(0)$ by condition (ii) and similarly $\mathcal{I}(-,1) \star D = D(1)$. Furthermore

$$\begin{array}{ccc} \mathcal{I}(-,0) \star D & = & D(0) \\ {\scriptstyle \mathcal{I}(-,g)\star D} \downarrow \Rightarrow \downarrow {\scriptstyle \mathcal{I}(-,f)\star D} & & {\scriptstyle g} \downarrow \Rightarrow \downarrow {\scriptstyle f} \\ \mathcal{I}(-,1) \star D & = & D(1) \end{array}$$

where the 2-cells are $\mathcal{I}(-,\gamma) \star D$ and $D\gamma$.

Define φ to be the coinverter

$$\mathcal{I}(-,0) \underset{\mathcal{I}(-,g)}{\overset{\mathcal{I}(-,f)}{\rightrightarrows}} \mathcal{I}(-,1) \xrightarrow{\;q\;} \varphi \qquad (\Downarrow \mathcal{I}(-,\gamma))$$

Now apply $(-) \star D$ to get the coinverter of $D\gamma$

$$D0 \underset{Dg}{\overset{Df}{\rightrightarrows}} D1 \xrightarrow{\;r\;} Q \qquad (\Downarrow D\gamma)$$

In fact, $\varphi\colon \mathcal{I}^{\mathrm{op}} \to \mathbf{Cat}$ is

$$0 \underset{g}{\overset{f}{\rightrightarrows}} 1 \; (\Downarrow\gamma) \quad \mapsto \quad \mathbb{I} \underset{1}{\overset{0}{\leftleftarrows}} \mathbb{1} \; (\Downarrow\cong)$$

More generally, if $\mathcal{L}$ is any 2-category, $D\colon \mathcal{I} \to \mathcal{L}$ a 2-functor, $\varphi \in [\mathcal{I}^{\mathrm{op}}, \mathbf{Cat}]$, the weighted colimit $\varphi \star D$ is characterized by a 2-natural isomorphism

$$\mathcal{L}(\varphi \star D, X) \cong [\mathcal{I}^{\mathrm{op}}, \mathbf{Cat}](\varphi, \mathcal{L}(D-, X))$$

2-natural in X. This is the general definition, but in the case where $\mathcal{L}$ is cocomplete the definition given above is really what this thing means. In a general 2-category, $(-) \star D$ will always be defined at the representables and cocontinuous in the weight insofar as it is defined.

I want to finish with an application to the formal theory of monads. First:

Kan extensions. Given 2-functors $\mathcal{L} \xrightarrow{F} \mathcal{M}$ and $\mathcal{L} \xrightarrow{G} \mathcal{K}$ with $\mathcal{M}$ small and $\mathcal{K}$ cocomplete the (pointwise[7]) **left Kan extension** $\mathrm{Lan}_F G\colon \mathcal{M} \to \mathcal{K}$ is defined by

$$(\mathrm{Lan}_F G)(M) = \mathcal{M}(F-, M) \star G.$$

Example. Take $\mathcal{L} = \bullet \rightrightarrows \bullet$, let $\mathcal{M}$ be the free living cofork $0 \rightrightarrows 1 \to 2$, and take $F\colon \mathcal{L} \to \mathcal{M}$ to be the obvious inclusion. Given a diagram $G\colon \mathcal{L} \to \mathcal{K}$, then $\mathrm{Lan}_F G\colon \mathcal{M} \to \mathcal{K}$ is given by $0 \mapsto G0$, $1 \mapsto G1$, and $2 \mapsto$ the coequalizer of $G0 \rightrightarrows G1$.

Dually, we have right Kan extensions. Taking $\mathcal{M} = -1 \to 0 \rightrightarrows 1$ the free living fork and F the obvious inclusion, then $\mathrm{Ran}_F G$ picks out the equalizer of $G0 \rightrightarrows G1$.

[7]This term is first used in the context of enriched categories in Dubuc's thesis.

The formal theory of monads. Recall $\mathbf{Mnd} = \Sigma\mathbb{A}_+$ is the 2-category classifying monads in a 2-category, and **Adj** is the 2-category classifying adjunctions in a 2-category. Given a monad in a 2-category $\mathbf{Mnd} \xrightarrow{G} \mathcal{K}$ is there an adjunction giving rise to this monad? We know two answers when $\mathcal{K} = \mathbf{Cat}$, the Eilenberg-Moore and Kleisli constructions. In general, we give ourselves the liberty of assuming that $\mathcal{K}$ is complete or cocomplete in which case we can form the right or left Kan extensions

$$\begin{array}{ccc} \mathbf{Mnd} & \xrightarrow{G} & \mathcal{K} \\ {\scriptstyle F}\big\downarrow & \nearrow & \\ \mathbf{Adj} & & \end{array}$$

If $\mathrm{Ran}_F G$ exists then it picks out

$$GA \underset{\longrightarrow}{\overset{\longleftarrow}{\scriptstyle\top}} V$$

where GA is the object with the endmorphism $G1 \colon GA \to GA$ that is a monad. (Recall the objects of $\mathbb{A}_+$ are natural numbers $0, 1, \ldots$.) Furthermore, this V is the **Eilenberg-Moore object** on GA.

In **Cat** the Eilenberg-Moore object is the usual category of $G1$-algebras. In $\mathcal{K}$, it's defined by a 2-natural isomorphism

$$\mathcal{K}(X, V) \cong \mathcal{K}(X, GA)^{\mathcal{K}(X,G1)}$$

where the thing on the right is the category of $\mathcal{K}(X, G1)$-algebras.

In general the way to define limit notions in a 2-category is to say that homing into that object gives the corresponding limit notion in **Cat**.

2. Interlude

Strict monoidal categories are just one-object 2-categories; monoidal categories are just one-object bicategories. One thing that's useful in dealing with these sorts of things is string notation.

String notation. The way this works is you take the Poincare dual of a pasting diagram in a 2-category or more generally in a bicategory: 0-cells become regions, 1-cells remain 1-cells (strings) but pointing in the opposite direction, and 2-cells become beads on the strings.

More precisely, we can display composite 2-cells in a 2-category/bicategory using string notation:

- regions of the page are labelled by objects
- lines on the page denote 1-cells, their domain and codomain being the adjacent regions
- nodes represent 2-cells

[Pictures omitted.]

A nice thing about string diagrams is it gives a very efficient way to specify adjoints. Given an adjunction $A \underset{g}{\overset{f}{\rightleftarrows}} B$ (with $\perp$), we have η and ϵ the unit and counit. In strings this looks like a cap and a cup (the beads are traditionally omitted) and the triangle identities say that the two "S" shaped diagrams formed from these can be "straightened," i.e., are equal to the straight lines corresponding to the identity 2-cells. Note, you don't need to label the regions because they can be reconstructed from the labels on the strings.

Exercise. Express the mates correspondence using string diagrams.

Note for monoidal categories, interpreted as one-object bicategories, you'll never have to label the region.

3. Monoidal categories

I'll end with a mixed bag of observations on monoidal categories, particularly concerning the Eckmann-Hilton argument.

Eckmann-Hilton argument. A monoid object in monoids is a commutative monoid.

There are analogs in 2-category theory:

2-d Eckmann-Hilton. A pseudomonoid object in (monoidal categories and strong monoidal functors) "is"[8] a braided monoidal category.

I.e., a pseudomonoid object in monoidal categories is a monoidal category $(C, \otimes, I)$ with a functor $\odot\colon C \times C \to C$ strong monoidal with respect to $\otimes$ and $J\colon \mathbb{1} \to C$ also strong monoidal with respect to $\otimes$. The coherence constraints α', λ', ρ' for $(\odot, J)$ are strong monoidal transformations with respect to $\otimes$.

It then follows that $I \cong J$, $\otimes \cong \odot$, and $\otimes \cong \odot^{\mathrm{rev}}$. From this we induce a natural family of isomorphism $A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A$ satisfying coherence axioms—the basic data for a **braided monoidal category**.

The key point with a braiding is that $\sigma_{A,B}$ and $\sigma_{B,A}$ need not be each other's inverses; if they are, this is called a **symmetric monoidal category**. In string notation, the convention is to draw $\sigma_{A,B}$ as two strings labelled A and B with the latter crossing under the former.

Example. Vector spaces form a monoidal category. A **bialgebra** in this context is an algebra and a coalgebra in which the coalgebra structure maps are algebra maps. An example is a group ring. For this to make sense, we need a little bit of additional structure on the monoidal category and that's the structure of a braiding.

A failure of Eckmann-Hilton. We can look at pseudomonoid objects in (monoidal categories and lax monoidal functors) in which case something quite interesting happens: A **2-monoidal/duoidal** category is a pseudomonoid object in monoidal categories and lax monoidal functors—i.e., $(C, \otimes, I)$ a monoidal category equipped with a second monoidal structure $(C, \odot, J)$ so that $\odot\colon C \times C \to C$ and $J\colon \mathbb{1} \to C$ are lax monoidal with respect to $\otimes$ and similarly for the coherence constraints of $(\odot, J)$.

This means we have two monoidal structures and maps

$$(A \odot B) \otimes (C \odot D) \to (A \otimes C) \odot (B \otimes D).$$

In the strong monoidal world these maps would be invertible. Setting B and C to be units we'd see that $\otimes \cong \odot$; setting C and D to be units we'd see these tensor products are braided. We also have maps

$$I \to J \qquad I \to I \odot I \qquad J \otimes J \to J.$$

Why is this structure interesting? It's because we see it quite a lot. First, a few observations:

- An equivalent definition: $(C, \otimes, I)$ is monoidal and $\otimes$, I, and their coherence conditions are oplax monoidal with respect to $(\odot, J)$.
- any braided monoidal category is a (2-monoidal/duoidal) category with $\otimes = \odot$, $I = J$. (Proof: a pseudomonoid in strong things is a pseudomonoid in lax things.)

[8]Not quite.

Duoidal categories are a natural setting for defining bialgebras.

Bialgebras in duoidal categories. Recall a bialgebra in vector spaces is an algebra that is also a coalgebra and moreover such that the coalgebra structure maps are algebra homomorphisms. For this to make sense, we need that if A is an algebra there is an algebra structure on $A \otimes A$, defined using the braiding. But we're not really using the fact that it's a braiding; we're just using the fact that it's a duoidal category. The point is in any duoidal category the tensor product $\odot$ lifts to the category of $\otimes$-monoids.

If $(C, \otimes, I, \odot, J)$ is a duoidal category, then it's a pseudomonoid in $\mathbf{MonCat}_{lax}$. There's a 2-functor $\mathbf{MonCat}_{lax} \to \mathbf{Cat}$ that sends $(C, \otimes, I)$ to $\mathbf{Mon}_{\otimes}(C)$ which sends pseudomonoids to pseudomonoids. Thus, $(C, \otimes, I)$, an object of $\mathbf{MonCat}_{lax}$, considered with the pseudomonoid structure $((C, \otimes, I), \odot, J)$, gets sent to some $(\mathbf{Mon}_{\otimes}(C), \odot, J)$. I.e., if C is duoidal, then the $\odot$-tensor lifts to $\mathbf{Mon}_{\otimes}(C)$.

Explicitly, $(A \otimes A \xrightarrow{m} A \xleftarrow{i} I) \odot (B \otimes B \xrightarrow{n} B \xleftarrow{j} I)$ is defined to be

$$(A \odot B) \otimes (A \odot B) \to (A \otimes A) \odot (B \otimes B) \xrightarrow{m \odot n} A \odot B \xleftarrow{i \odot j} I \odot I \leftarrow I.$$

Dually, the $\otimes$ monoidal structure lifts to $\mathbf{Comon}_{\odot}(C)$.

Definition. A **bialgebra** in a duoidal category C is an object of $\mathbf{Comon}_{\odot}(\mathbf{Mon}_{\otimes}(C)) \cong \mathbf{Mon}_{\otimes}(\mathbf{Comon}_{\odot}(C))$.

Explicitly, this is

- an object $X \in C$
- a monoid structure $X \otimes X \xrightarrow{m} X \xleftarrow{i} I$
- a comonoid structure $X \odot X \xleftarrow{c} X \xrightarrow{u} J$
- axioms, most importantly that $X \otimes X \xrightarrow{m} X \xrightarrow{c} X \odot X$ equals the composite

$$X \otimes X \xrightarrow{c \otimes c} (X \odot X) \otimes (X \odot X) \to (X \otimes X) \odot (X \otimes X) \xrightarrow{m \odot m} X \odot X.$$

Recall a braided monoidal category is a special case of this in which the two tensor products are the same. In some ways, the axioms are even clearer from the duoidal perspective, in particular the need for the swapping over in the middle, which is necessary for type checking in the duoidal setting but isn't obviously so in the braided setting.

I haven't really given any examples, but I'll leave that to Marcelo on Friday.

References

[KS] G. M. Kelly and R. Street. A review of the elements of 2-categories. LNM 420.

[SS] S. Schanuel and R. Street. The free adjunction. Cahiers de topologie et géométrie différentielle catégoriques.