Enrico Vitale

Université catholique de Louvain

Another snake lemma

The classical snake lemma states that, if

$$\begin{array}{ccccc} \mathrm{Ker}[p] & \longrightarrow & A & \xrightarrow{p} & B \\ {\scriptstyle f}\downarrow & (1) & {\scriptstyle g}\downarrow & (2) & \downarrow{\scriptstyle h} \\ \mathrm{Ker}[q] & \longrightarrow & C & \xrightarrow[q]{} & D \end{array}$$

is a commutative diagram in the category Ab of abelian groups and if p is surjective, then there exists an exact sequence

$$\mathrm{Ker}[f] \to \mathrm{Ker}[g] \to \mathrm{Ker}[h] \to \mathrm{Coker}[f] \to \mathrm{Coker}[g] \to \mathrm{Coker}[h].$$

In my talk, I will show that, if we drop the assumption that p is surjective, we still have an exact sequence

$$\mathrm{Ker}[f] \to \mathrm{Ker}[g] \to \mathrm{Ker}[h] \to \mathrm{Coker}[\langle g, p \rangle] \to \mathrm{Coker}[g] \to \mathrm{Coker}[h],$$

where $\langle g, p \rangle$ is the factorization of (2) through the pullback of q and h. Moreover, $\mathrm{Coker}[f]$ always embeds into $\mathrm{Coker}[\langle g, p \rangle]$, and is isomorphic to $\mathrm{Coker}[\langle g, p \rangle]$ if p is surjective.

To understand the new sequence, we have to look at it from the point of view of 2-categories: the old sequence is associated with the kernel (1) of the morphism (p, q) in the category $Ab^{\to}$, whilst the new sequence is associated with the kernel of the morphism (p, q) in the 2-category $Ab^{\to}$.

Everything remains true if we replace Ab by any abelian category or, under suitable conditions on g and h, if we replace Ab by any pointed, regular and protomodular category.