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Codensity and the ultrafilter monad

Even a functor without an adjoint induces a monad, namely, its codensity monad. (It is defined subject only to the existence of certain limits.) I will expand on an undeservedly ignored theorem of Kennison and Gildenhuys: that the codensity monad of the inclusion of (finite sets) into (sets) is the ultrafilter monad. This result is analogous to the correspondence between measures and integrals. So, for example, we can speak of integration against an ultrafilter.

A related fact is that the codensity monad of the inclusion of (finite-dimensional vector spaces) into (vector spaces) is double dualization. From this it follows that compact Hausdorff spaces have a linear analogue: the 'linearly compact' vector spaces of Lefschetz.

The same technique also shows how ultraproducts, important in model theory, are categorically inevitable: the codensity monad of the inclusion of (finite families of sets) into (families of sets) is the ultraproduct monad.